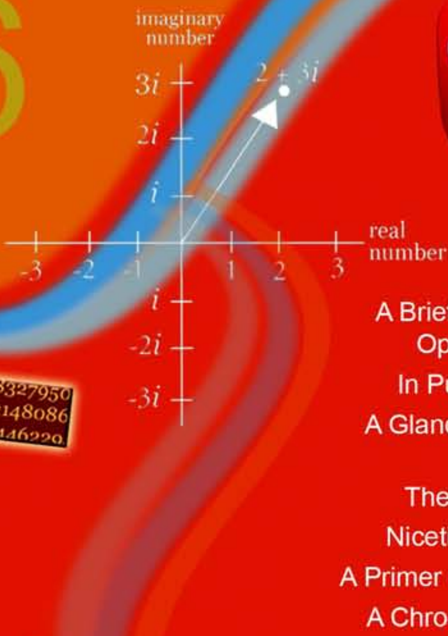
Moments in Mathematics



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Srinivasa Ramanujan

Rintu Nath

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Preface

Mathematics is an integral part of our life. We encounter mathematics in different forms in our everyday life. It also plays a predominant role in the overall development of the society. An understanding and appreciation of mathematics is therefore an essential life skill. While it aids in solving many real-life problems, it also helps us develop a logical thinking process.

Mathematics is not about a monotonous and complex process of measuring, counting and accounting. Mathematics is also not about remembering complex rules to solve problems in the examination. Mathematics is to develop our logical ability to question, investigate, and explore solutions to many real-life problems. Mathematics is perhaps the only tool that helps us in developing and refining such aptitudes.

Googol, a young boy, always wonders about many topics on mathematics. He quizzes his mathematician uncle with his queries. His uncle explains the intricacies of mathematical issues and motivates him to ask more questions. The conversation between Googol and his uncle unfolds the mystical and the wonderful world of mathematics.

When uncle tells Googol that zero is a mysterious number that is neither even, nor odd; neither prime nor composite; it captures the attention of Googol. His conversation with his uncle enlightens him about many other fascinating facts about zero like the factorial of zero or the impossible scenario of division by zero, and many more.

Uncle takes Googol to the wonderful world of numbers. It is a fascinating experience for him to know about the irrational, transcendental and algebraic numbers. Googol learns why the π is a transcendental number.

Uncle elaborates on the mysterious and esoteric constants like *pi* (π), the Euler's number and the golden ratio. The decimal representation of these numbers never repeats and never ends. Several billion digits of pi have been calculated using super computers. The beauty of these constants captivates Googol. He wants to know more about these amazing constants, their mathematical interpretation and their applications in daily life.

Googol learns that the imaginary number is essentially a part of the complex number and it has an important role in solving many mathematical problems. The term 'complex number' does not mean that it is an intricate or complicated topic in mathematics. A complex number is formed by using real and imaginary numbers together.

Mathematicians have been asking questions about prime numbers for more than twenty-five centuries, and every answer seems to generate a flurry of new questions. Googol is thrilled to know that most of the unsolved problems in mathematics are related to prime numbers.

The binary number system forms the basis for the operation of computers and all digital circuits. Any number can be represented in the binary number system using different combinations of two numeric symbols, 0 and 1. Uncle explains to Googol how the binary number system enables computers to represent and interpret information using electrical signals.

Googol is surprised to know that there is a 97% chance that two friends in a class of fifty will have the same birthday. While solving this mystery, uncle elucidates the rules of probability, citing examples of the tossing of coin, throwing of dice and playing cards.

Uncle clarifies that the calculus is the study of change and how the differential and integral calculus are essential in solving many real-life problems, which otherwise could have been intractable.

Googol feels proud of Indian mathematicians like Aryabhata, Brahmagupta, Madhava, Ramanujan and many others for their seminal contributions to the development of many mathematical principles.

This book will take you to the beautiful and mesmerising world of mathematics. Explore this world through the inquisitive eyes of Googol.

Rintu Nath

A Brief History of Zero

'Googol, can you say what is common in duck, egg and love?'

The question came from my uncle. I was doing my math homework and he was absorbed with some intricate problems in mathematics when suddenly he popped the question to me.

I fumbled for a second. I did not have a clue about the answer.

'Do you want more clues?' uncle asked me again seeing my blank look.

'Well, yes…' I was not sure how much that would help.

'Well, here is a cryptic clue for you: *number delivered in a circular letter*,' said he.

'I suppose all letters delivered by postman are rectangular. I did not see a circular letter ever,' I tried to reason with him.

'Fool, the word *letter* is a pun'.

This time uncle was seemingly upset over my hurried reply without giving much thought to it.

Well, before you also try thinking with me, let me introduce myself first. I am Googol. Of course, this is my nickname, but I like the name very much. And everybody calls me by this name. When I was born, my mathematician uncle gave this name to me.

My uncle later told me that the name *googol* carries an interesting story. In 1938, Dr. Edward Kasner (1878-1955), a mathematician, asked his nephew Milton Sirotta, then nine years old, to think a name for a really big number, namely, 1 with a hundred zeros after it (10^{100}). Milton came up with the name *googol*. Then, at the same time, to name a still larger number, Dr. Kasner coined the term *googolplex*. It was first suggested that a *googolplex* should be 1, followed by writing zeros until you got tired. This was a description of what would happen if one actually tried to write a *googolplex*, but as you can presume that different people got tired at different times. The *googolplex*, then, is determined as a specific finite number, with so many zeros after the 1 that the number of zeros is a *googol* (10^{googol}). A *googolplex* is much bigger than a *googol*; much bigger even than a *googol* times a *googol*. These inventions caught the public's fancy and are often mentioned in discussions of very large numbers. In this context, let me give you another bit of information that Dr. Edward Kasner wrote a book with James Newman titled *Mathematics and the Imagination*.

Now about my uncle's riddle. I tried to get the information from the cryptic clue. The clue that that word *letter* is a pun led me to think about our alphabetic letter. And here we have the circular letter 'O' and the number delivered with that letter is... 'Oh, I got that!' I exclaimed, 'the answer is Zero'.

But still I was not sure about how to relate zero with *duck*, *egg* and *love*. So I commented, 'But uncle, how are other three words related with zero?'

'Well, you know when a cricketer gets a *duck*…'

'Yes, when he scores no run that means zero.'

'And in tennis or badminton, you might have heard the score as 10-*love*.'

'And in that case also the score *love* means zero.'

'The French word for *egg* is *l'oeuf*. Now since *zero* looks more or less similar in shape as that of an egg, so *l'oeuf* after some changes became *love*, which the present reason of calling a zero as *love*.'

'There are of course a lot of names given to *zero* or something conceptually as zero like *cipher*, *aught*, *nought*, *naught*, *not*, *nil*, *null*, *nothing*, *none*.'

'And I have heard people say the letter 'O' to say zero like O-1-3-1 to represent 0131.'

'Yes, you are right. Sometime it is quicker and easier to pronounce monosyllable words. That may be the reason for speaking 'O' as zero. Of course, there are some incidences where something like 'O' was used by early mathematicians to represent zero.'

'He must have been a genius who discovered zero!'

'Indeed he was. But there is a long history of zero…'

'Tell me something about it,' I was very eager to know.

'Initially, the zero as a number was not available. There was the idea of empty space, which may be thought of conceptually as similar to zero. Babylonians around 700 BC used three hooks to denote an empty place in the positional notation. They used a symbol sort of like a “Y" for one, and a symbol sort of like “<" for ten.'

'What about Greek mathematicians?' I asked.

‘Yes, almost during the same time, Greek mathematicians made some unique contributions to mathematics. The interesting feature is that Greek mathematics is mostly based on geometry. Euclid wrote a book on number theory named *Elements*, but that was completely based on geometry. The newer system of Greek numerals is known as *Alexandrian numerals*. It is more than 2000 years old and used Greek letters for 1 to 9, 10 to 90, and 100 to 900. 1 was written as ‘A’ (alpha), 10 as ‘I’ (iota), and 100 as ‘ρ’ (rho). They did use a limited place system, so ‘111’ was written as ‘ρIA’. For 1000 and above they used a mark such as ‘,’ or ‘/’ before the number of thousands. So, ‘1000’ is ‘,A’ or ‘/A’ , and ten thousand is ‘,I’ or ‘/I’. 'So there was no concept of zero even for Greek mathematicians,' I wondered.

'Not exactly like that. Greek astronomers might have felt the need for empty space and began to use the symbol 'O'. It is not clear why they favoured the particular notation. It may be related with the first letter of the Greek word for nothing namely *ouden* or it may come from *obol*, a coin of almost no value.'

'I think the Romans also did not have any idea of zero, since I know Roman number system has letters, like 'X' for 10,' I said.

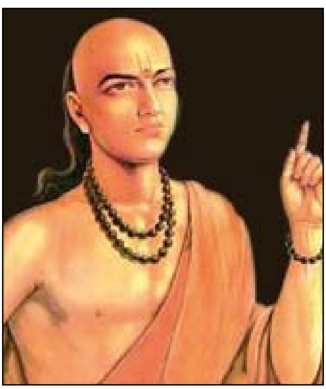
'You are right. Roman numerals for 1, 10, 100, and 1000 are I, X, C, and M. It is interesting that Greeks or Romans relied more on the Abacus that they used to perform arithmetic operations such as addition, subtraction, division, or multiplication and they may not have thought of any operation related with zero.'

'So zero was not there in the mind of those early Greek or Roman mathematicians,' I said.

'Yes, in early history of most of these civilisations, there is no concrete evidence of zero or its use. This may be due to conceptual difficulty to figure out something, which would represent nothingness.'

'What about Indian civilisation?' I got interested.

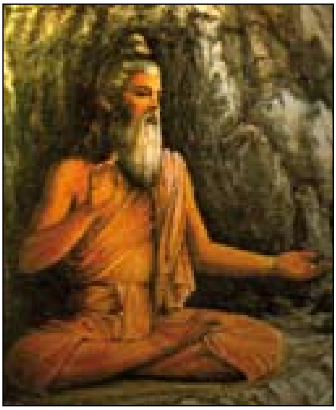
'Around AD 650, the use of zero as a number came into Indian mathematics. The Indians used a place-value system and zero was used to denote an empty place. In fact there is evidence of an empty placeholder in positional numbers from as early as AD 200 in India. Around AD 500 Aryabhata devised a number system, which had no zero as a positional system, but used it to denote empty space. There is evidence that a dot had been used in earlier Indian manuscripts to denote an empty place in positional notation. For example, to represent '100' it would be two dots after 1.'



Aryabhata 500 AD

'So use of zero as number started,' I said.

'In AD 628, Brahmagupta wrote *Brahmasphutasiddhanta* (The Opening of the Universe), and attempted to give the rules



Brahmagupta 628 AD

for arithmetic involving zero and negative numbers. He explained that given a number, if you subtract it from itself you obtain zero. He gave the following rules for addition, which involve zero: *The sum of zero and a negative number is negative, the sum of a positive number and zero is positive; the sum of zero and zero is zero.* Similarly, he gave the correct rules for subtraction also.

'Brahmagupta then said that any number when multiplied by zero is zero, but when it comes to division by zero, he gave some rules that were not correct. But remember, when the concept was just developing, it is quite usual that he would make mistakes. So it was an excellent attempt to visualise number system in the light of negative numbers, zero and positive numbers.'

'Brahmagupta seems to be a genius!' I exclaimed.

'In AD 830, Mahavira wrote *Ganita Sara Samgraha* (Collections of Mathematics Briefings), which was designed as an update of Brahmagupta's book. He correctly stated the multiplication rules for zero, but again gave incorrect rule for division by zero.'

'So could anybody make the correction?' I said.

'After 500 years of Brahmagupta, Bhaskara tried to solve the problem of division by stating that any number divided by zero as infinity. Well, conceptually though it is still incorrect, but Bhaskara did correctly state other properties of zero, such as square of zero is zero and square root of zero is also zero.'

'So Indian mathematicians developed the concept of zero and stated different mathematical operations involved with zero. But how did the concept spread to all over the world?' I asked.

What's in a name?

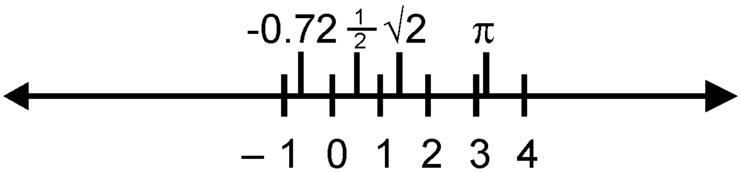
Portuguese	:	zero
Italian	:	nullità
French	:	zéro
German	:	null
Spanish	:	cero
Danish, Indonesian	:	nol
Dutch	:	nul
Finnish	:	nolla
Hungarian	:	zero
Norwegian	:	null
Swedish	:	noll

- Words similar or closer to meaning of *zero* are *cipher*, *aught*, *nought*, *naught*, *not*, *nil*, *null*, *nothing*, *none*.
- Probably the synonymous word (*z)ero* and (*n)il* produced the word *zilch*, which is a slang meaning nothing. Sometime a person is also called *zilch* to indicate as being insignificant or nonentity.
- The word *goose egg* is another slang for *zero*, especially when written as a numeral to indicate that no points have been scored.
- In mathematics, the terminology *infinitesimal* indicates a function or variable continuously approaching zero as a limit.
- *Nilpotent* is an algebraic quantity that when raised to a certain power equals zero.

'The Islamic and Arabic mathematicians took the ideas of the Indian mathematicians to further west. Al-Khwarizmi described the Indian place-value system of numerals based on zero and other numerals. Ibn Ezra, in the 12th century, wrote *The Book of the Number*, which spread the concepts of the Indian numeral symbols and decimal fractions to Europe.

'In 1247 the Chinese mathematician Ch'in Chiu-Shao wrote *Mathematical Treatise in Nine Sections*, which used the symbol 'O' for zero. In 1303, Chu Shih-Chieh wrote *Jade Mirror of the Four Elements*, which again used the symbol 'O' for zero.

'In around 1200, Leonardo Fibonacci wrote *Liber Abaci* where he described the nine Indian symbols together with the sign '0'. However, the concept of zero took some time for acceptance. It is only around 1600 that zero began to come into widespread use after encountering a lot of support and criticism from mathematicians of the world.'



'So *shunyam* given by our forefathers was recognised in the world and made its place permanently as zero,' I commented.

'Interestingly, the word zero probably came from the Sanskrit word for *shunyam* or the Hindi equivalent of *shunya*. The word *shunyam* was translated to Arabic as *al-sifer*. Fibonacci mentioned it as *cifra* from which we have obtained our present *cipher*, meaning empty space. From this original Italian word or from alteration of Medieval Latin *zephirum*, the present word zero might have originated.'

'That's really interesting. Uncle, I have a question. I have still a dilemma regarding division with zero. Could you please clarify more?' I expressed my problem.

'Well dear, it will take some more time for clarification. I will take it on some other day,' uncle remarked and again became engrossed with his problem after this long discussion.

I had also to finish my homework, so I stopped for the time being. But zero was moving in my brain, and many questions started coming in my mind regarding this amazing concept of *nothing*.

Operation Zero

I was celebrating *Holi* with my friends and relatives. It was colourful everywhere. The beautiful spectrums of colours of *gulal* made everything picturesque. It looked as if a little child played with his brushes quite liberally in his canvas.

My uncle was also in festive mood but he does not like all those colours. So he was just watching us from a distance and sometime cheering us up saying 'Attention Googol, someone is coming in this way!'

At one o'clock in the afternoon, I washed off the colours and had my bath. As I returned to drawing room, I found that my uncle was relaxing and watching television.

'So you had a nice time, Googol,' he said without taking his eyes off the television.

'Oh yes, it was very enjoying,' I replied and came near to him.

'But I did not play *Holi* with you! So I think that I can do it now,' he said, and suddenly rubbed something in my face. 'Uncle, I just had my bath and you …' I protested.

'Cool down, dear! Check yourself in the mirror, Googol, ' he said.

I realised that he had just put some talcum powder in my face.

'Don't you think that we should celebrate *Holi* with white colour only?' the question was directed at me.

'I could not get that,' I said.

'What physics say…,' he implied.

'Oh yes, white colour is the mixture of all colours,' I said and spread the powder over my face.

'Right, so the festival of colours will be really meaningful with white colour,' he added.

'I got that. That means if I put black colour in your face, actually I will not put any colour,' I quipped, 'you know, physics say that…'

'You naughty boy!' smiled uncle.

'So before putting forward your suggestion on real colourful *Holi*, we have to prepare being painted with *nothing* in our face from scientifically inclined persons,' I said.

'Well, I admit,' uncle said, 'but while dealing with *nothing* in mathematical world, you should be more cautious.'

'Yes, I remember, you earlier told me about the problems related with operation of zero. Could you please clarify more?' I was very eager to hear the next part of story and so I drew myself closer to him.

'Well, do you know about real numbers?' uncle asked.

'Real numbers consist all rational (i.e the numbers which can be express as *p/q,* like 2) and irrational numbers (which cannot be expressed as fraction, like √2),' I said.

'Right. Now all these real numbers can be placed uniquely in a real line towards both positive and negative direction. Hence all positive, negative, even, odd, rational and irrational numbers correspond to only a single point on the line,' uncle explained.

I nodded my head understanding the real line principle.

'Could you now tell me, where zero stands in this real line?' uncle asked me.

'It seems that it is just standing as borderline between the positive and negative numbers,' I replied.

'Yes, among these real numbers, zero has the most important and unique position. It is in the intersection between positive and negative numbers. If you go to the right side from zero, it is positive numbers and if you go towards the left side of zero, it is all negative numbers. So essentially zero is neither positive nor negative number, it is the borderline for positive and negative numbers, or it is neutral in that sense. In fact this is the only number in the real number world, that is neither positive nor negative.'

'So zero is a lonely person standing in the borderline with nobody around it to share its characteristics – even 1 is not such a lonely number,' I joked.

'To be precise, zero as single entity has no power of its own. Even if you put the poor fellow to the left side of any number (without any decimal), still it is powerless. But if you start adding it to the right side of a number, then zero starts showing its power and the number increases by ten times for each addition.'

'So a lonely and tiny person can be real powerful depending on the situation. But does zero share any feature of an odd or even number?' I questioned.

'Well, simply speaking an even numbers are those which are divisible by 2 and odd numbers are those that are not divisible by 2. Since theoretically zero is divisible by 2, so zero is considered to be even number. But many people do not consider zero as even number since zero is divisible by any number irrespective of positive and negative and divisibility with 2 is not very unique feature to zero as that of other even numbers.'

'What about zero as prime number?' I got interest in the discussion.

'A prime number is a positive integer that has no positive integer divisors other than 1 and the number itself. So by definition, prime number is a positive integer and should be placed in the right hand side of 1 in our real line scale. Clearly zero does not fit in this definition and so zero is not a prime number.'

'I can understand now that Brahmagupta might have to give a lot of thinking to define zero in a number system and to present the rules for its operation. I can remember that he correctly defined the position of zero in the number system and gave the rules of addition, subtraction and multiplication.'

'You are absolutely right. If we add zero with a positive and negative number, then we will remain in the same number point in the real line scale. And if we multiply any positive and negative real number with zero, then we will be directed straight to the position zero.'

'And what about the division by zero?' I asked.

'Well, the division by zero is a tricky one. Brahmagupta himself could not describe the operation properly and later Bhaskara also mentioned it incorrectly.'

'I can remember what Bhaskara said: if any number is divided by zero, it is infinity.'

'Well, at first instance, assigning some positive number divided by zero as infinity or very high value, seems logical. For example, if you continue to divide a real positive number by a smaller number, then your result will go on increasing. Like:

$10/10 = 1$

$10/1 = 10$

$10/0.01 = 1000$

$10/0.0001 = 10,0000$

:

:

$10/10^{-99} = 10^{100}$

and so on'

'So when we will divide the number by zero, it will go towards infinity or a very very high value,' I said.

'Well, as you divide by a smaller number and go towards zero, the result increases. But remember, still the smaller number is not equal to zero. Therefore, you are *not* actually doing any division by zero, rather you are predicting a trend, which might be possible if divisor reaches a value, closer to zero or very small numbers. But whatever the smallest number you can think of, another number smaller than that exists. Moreover, you should remember that infinity is a concept, an abstract thing, not a number as defined in our number system and all rules of mathematics are invalid while you will consider operation with infinity. Like if you add infinity and infinity you will not get twice the value of infinity. It is still infinity!'

'Then it is wrong to say that a number divided by zero is infinity,' I said.

'Exactly! In fact, in the very first place it is wrong to attempt to divide a number by zero,' uncle emphasized.

'So what should be the actual explanation for this situation?' I was curious.

'Well, let me give you a further clarification. A division is essentially the inverse of multiplication rule. That means if you divide 10 by 2, then you will get 5. And if you multiply 5 with 2, then you will get your original value back again. Through algebra, we can put it like this:

If $(a / b) = c$, then $a = (b * c)$

Let's see what will happen if we follow the infinity theory. Assume that $a = 10$ and $b = 0$. Now, if you attempt to do (a / b) and assume c = infinity, then according to rule of multiplication, we get 10 = (0 * infinity). But the rule of multiplication for zero says that anything multiplied with zero is zero. That means, applying the multiplication rule in right hand side gives us finally: 10 = 0. So you cannot get back 10 by multiplying the elements in the right hand side, rather you will get some absurd result as above while attempting and evaluating something divided by zero.'

'So we should not divide a number by zero…'

'Yes! The uniqueness of division breaks down when you attempt to divide any number by zero since you cannot recover the original number by the inverting the process of multiplication. And zero is the only number with this property and so division by zero is *undefined* for real numbers. So you should *never* attempt to do a division with zero. In fact, it is meaningless to attempt to do this operation.'

'Ok, I should not attempt to do any mathematical operation related with division by zero since it is not even defined in our mathematical world.'

'Let me give you another very common example to show what could happen if you ever try to attempt to do something like that.

Let, $x = y$

Multiplying both side by y, we get,

$x * y = y * y$

If we subtract y^2 from both side, then it becomes:

$x *y - y^2 = y *y - y^2$

This can be written as

$x*y - y^2 = y^2 - y^2$

With some simple algebra, the expression becomes:

$y * (x - y) = (y + y) * (y - y)$

Since we have assumed, $x = y$, so we can write: $y (y - y) = (y + y) * (y - y)$

Now if we divide both side by $(y - y)$, then it comes as: $y = 2y$

Or if we cancel out y from both sides, it is $1 = 2$.

Ok Googol, tell me now why does this type of meaningless result come after doing all those seemingly legal algebraic operation?'

'I think that the cancelling out $(y - y)$ is not the correct method…'

'Right! You can see that we are actually cancelling out $(y - y)$ from both side, which actually equals to zero and legally we cannot do the simple division with zero and if you do, it will make thousands of mathematical rules invalid. Simply speaking, there is no number in real number world, which equals to the expression: *x divided by zero*.'

'Now I can understand, that is why the division by zero is made *undefined* in mathematical terminology so that if we follow this simple single rule then we don't have to worry about thousands of other mathematical rules which will be valid always.'

'Yes! This is the reason that in all computer programs or mathematical calculations, one should take care of this vital operation and there should have appropriate strategy to deal with this situation. Imagine, a remotely controlled rocket is going towards a distant star and the computer installed in it, is doing millions of vital calculation every second. But the scientists who programmed the computer just inadvertently forgot to tell the computer what it should do if something like division by zero occurs. And unfortunately if it occurs, the computer will stop working and it will wonder what to do with this undefined operation. So all the efforts of the scientists will be a waste! Zero is so powerful.'

'I have seen that if I try to do the division by zero in calculator it shows 'E'.'

'Right. This means the operation you are attempting is erroneous and you should not attempt this operation.'

'Ok, so something divided by zero is undefined and it is wrong to do any operation involving that. Is this rule only applicable to real numbers?'

'Well, this is true for the world of real numbers. But in calculus theorem, limits involving division by a real quantity, which approaches zero, may be well-defined. For example, you will get the expression like this: *limit x tends to zero* (*sin x* / *x*) equals to 1, i.e $\lim_{x \to 0} \left(\frac{\sin x}{x}\right) = 1$. But be careful, our concept of something divided by zero as undefined still holds good, since in above function you are *not* attempting any value of *x* which is equal to zero. For the same reason limits like *limit x tends to zero* (1 / *x*) i.e $\lim_{x \to 0} \left(\frac{1}{x}\right)$ do not exist.'

'When it is so critical phenomenon with something divided by zero, I wonder what about zero divided by zero?'

'Well, this is another interesting case. Mathematically speaking, an expression like zero divided by zero is called *indeterminate*. To put it simply, this is a sort of expression, which cannot be determined accurately. If you see the expression properly, you can't assign any value to it. That means (0 /

0) can be equal to 10, 100 or anything else and interestingly the rule of multiplication also holds true here since 10 or 100 multiplied by zero will give the product as zero. So the basic problem is that we cannot determine the exact or precise value for this expression. That's why mathematically (0 / 0) is said to be *indeterminate*.'

'It's amazing! It is now understandable that why our forefathers had the problem in defining operations involved with zero. They have done really a great job. I remember, Bhaskara has also given the correct rule for square of zero.' 'Yes. The square of zero is similar in meaning to multiplying zero two times. So according to multiplication rule, it should be zero. It is not only square, but cube and all powers of zero is zero.'

'And what about the square root of zero?'

'Similar to square or cube of zero, the square root, cube root, fourth root and so on all will be zero. You can easily get the logic if you think that the square root of zero should be such a number, which if multiplied twice should give you zero. Or in other way, square root is nothing but taking ½ as power and so all powers of zero are equal to zero.'

'And what will happen if I make zero as a power to some number?'

'Well, if you put zero as power to any number, it is always one. This comes from the rules by which we deal with operations involved with powers. For example:

$x^2 = x * x$

$x^{-2} = 1 / x^2 = 1 / (x * x)$

Hence, x^0 can be written something like: x^{2-2}

Which we can separate as: $x^2 * x^{-2}$

This gives us: x^2 / x^2, which makes our result as 1.'

'What about zero to the power zero?'

'Mathematically, this situation is similar to zero divided by zero. Using limit theorem, it can be found that as *x and a tend to zero*, the function a^x takes values between 0 and 1 inclusive.

So zero to the power zero is also termed as *indeterminate*. But modern day mathematicians are giving many new theories and insights regarding proper explanation of zero to the power zero. Some mathematicians say that accepting $0^0 = 1$ allows some formulas to be expressed simply while some others point out that $0^0 = 0$ makes the life more easier. So this expression is not as naïve as it looks like!'

'Now I know about two indeterminate forms in mathematics. The first one is (0 / 0) and the second is 0^0. Is there any other indeterminate form involving zero?'

'Well, to be precise there are seven indeterminate forms in mathematics involving 0, 1 and infinity.'

'I have recently did many permutations and combinations during World Cup matches. So I feel curious to know about the factorial zero.'

'The factorial of zero is equal to one. This is because the number of permutations you can do with zero elements is only one. This also can be proved mathematically. Remember here that the factorial of one is also one.'

'Uncle, frankly speaking I have discovered zero today in completely different perspective. Till now I used to think that zero is a tiny number and makes everything easy while it appears in calculations. But now I can understand that this tiny number zero could give mathematicians in the world so many troubles. Hence whenever it is operation with zero, it should always be handled with care and caution. Am I right?'

'Yes, you are absolutely right! Zero is tiny number, but you should never ignore its might. Imagine the world without zero. Not only mathematics, but all branches of sciences would have struggled for more clear definitions in their individual contexts, had zero not exist in our number system. Numbers from 2 to 9 are absent in binary system, and so are 8 and 9 in octal system. However, zero is everywhere and it is one of the significant discoveries of mankind. Thanks to the ingenuity of our forefathers.'

Undefined
In mathematics, an expression is said to be **undefined** which does not have meaning and so that is not assigned an interpretation. For example, division by zero is **undefined** in the field of real numbers.

Indeterminate
A mathematical expression is said to be **indeterminate** if it is not definitively or precisely determined. Certain expressions of limits are termed as **indeterminate** in limit theorem. There are seven indeterminate forms involving 0, 1 and infinity (∞).

$$(0 / 0),\ 0.\infty,\ (\infty/\infty),\ (\infty-\infty),\ 0^0,\ \infty^0,\ 1^\infty$$

Identically Zero
Sometime, to put it sufficiently strongly, a quantity that rigorously assumes the value of zero is said to be **identically zero**. A quantity that is identically zero is said to be **vanishing**, or sometimes to **vanish identically** as mentioned above.

Zero free
An integer value whose digits contain no zeros is said to be **zero free**. For example, square of 334 is a zero free square. In recent times a lot of interesting works are going on to find the **zero free** number for n^{th} power.

'I think that if any organization codes a task as *Operation Zero* then we can presume it may not be a simple task at all…'

'Yes, it should be really the most difficult task since mathematics presume that the task involve many undefined and indeterminate operations!'

'Uncle, I have an idea. To extend our analogy of numbers with colours, it seems more appropriate to assign zero as black colour. They are physically nothing, but both of them have tremendous impact while you see them with other colours or numbers.'

'That's a good analogy, Googol! Put your imagination in motion…'

In pursuit of π

I had a nice Easter celebration this time. All of our family went to our neighbour Samuel uncle's house in Easter Sunday to greet him and his family. To celebrate the occasion, Samuel uncle put a lot of food items in our disposal. There were eight or ten sweet dishes, apple pies, breads, cakes, four different tastes of ice creams and so on. I enjoyed very much the nice festive moods of the evening with those delicious foods and friendly gathering.

We returned home around 8 o' clock in the evening. I had already finished my homework for the next day's school. So I just prepared my next day's school bag and joined my uncle in the drawing room.

'I think that you've relished those sweet dishes, my dear Googol,' uncle smilingly said.

'Those were really nice, aren't they?' I was a bit ashamed.

'Indeed those were,' uncle said, 'so let me give you a riddle on food.'

'A riddle will be as good as those lovely foods,' I said.

'So here is the cryptic clue: *Perfect Food*,' uncle said.

'I understand all food items in the party were perfect,' I said.

'Well, no doubt that all food items were perfect, but *apple-pie* was the perfect food. Do you know why?'

'No, I don't have a clue.'

'I hope that there is no doubt that apple-pie is a *food* that you relished this evening.'

'But why it's a perfect one?'

'In informal English usage, the word *apple-pie* means something *perfect*. For example: *Put the books in the shelf in apple-pie order*.'

'Now I understand!'

'Well, this one is something related to the previous answer. This must be easier: *Take the end off chart number*.'

'We are talking about pie. So it must be related with pie *chart*.'

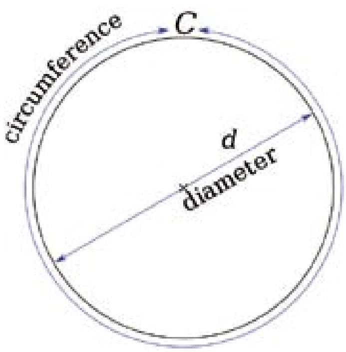
'So if you *take the end off* pie, what *number* do you get?'

'Yes, it is *pi*. I know it is a *number* represented by Greek symbol π and equals to 22/7.'

Uncle smiled at me and looked a bit disappointed hearing my quick reply.

'Dear Googol, if you say π equals to 22/7, you are in fact making a wrong statement.'

'So π is not equal to 22/7…'



'If you assume π equals to 22/7, then you are making an error of 0.0004 time of actual π value.'

'So what should be the value of π?' I wondered.

'Before coming into the actual value of π, tell me what geometrical concept is represented by π.'

'π is the ratio of circumference to the diameter of a circle.'

'Good. Or in other words also, you may put it as the ratio of area to the square of radius of circle. But interestingly long ago mathematicians used to think that these two values are different.'

'So there is a story behind the calculation of the value of π,' I got interested.

'Indeed, there is a long story,' uncle emphasized.

'Tell me something about this,' I could not resist hearing the story from my uncle.

'Well, the fact that ratio of circumference to the diameter of a circle is a constant was known from ages. However, the very first instance of mentioning something similar to π seems still a mystery. Most probably Egyptians mentioned about this constant in their writings in papyrus scroll as early as 1650 BC. Of course that time it was not mentioned as π as we do today, but they did mention about area of a circle using a rough estimate of a constant what we now say as π. There is good evidence that value as 256/81 (that is equivalent to 3.16) was used a value for this constant. Babylonians around the same time used 25 / 8, or decimal equivalent of 3.125 as the value of this constant. In Bible also, this constant is mentioned and its value is written as 3.

'In earlier times geometry was very advanced and the concept of π was originated from geometry itself. So somebody must have tried to calculate pi from geometrical concept.'

'Archimedes seemed to provide the first theoretical calculation of π around 200 BC. But once again let me remind you that though we are mentioning here the word π, you should note that it was not represented by that specific Greek symbol 'π ' till the beginning of 18^{th} century.'

'Ok, I understand that. But uncle, I have just a small query. Is this Archimedes was the same person with whom Archimedes principle in hydrostatics is related?'

'Yes. Archimedes was the man of many qualities. Apart from some excellent works in geometry, he devised many machines and developed theories in hydrostatics and number system.'

'So what was the approximation of pi according to Archimedes?'

'He said the constant takes the value between 223/71 and 22/7. The interesting thing is that he did not claim to know the exact value of π, rather he mentioned about the boundary of values between which π exists.'

'Archimedes mentioned that π is not equal to 22/7 such a long time ago!'

'Indeed he did so. Following Archimedes statement, if you take the average of the upper and lower boundary values, and convert to decimal points, then you will get the value as 3.1418, which is an error of about 0.0002 times of actual π value.'

'It seems that Archimedes was very close when nobody was sure about the value of π.'

'When Archimedes derived the above boundary, you should remember that there was no concept of algebra or trigonometry. Neither the decimal number system was in existence. So he used the pure geometry using the concepts of circle and regular polygon in deriving his expressions in term of fractions.'

'So it is really a great achievement!'

'In fact, historians found that no mathematician was able to improve over Archimedes' method for many centuries. A number of persons used his general method of polygonal measurement for more accurate approximations. Interestingly, Archimedes was also first to tell that both ratios that we have described for π (*i.e.*. 1. circumference to the diameter of circle and 2. area to square of radius of circle) indicate the same value.'

'Did any other mathematician improve this value?'

'The Greek astronomer Ptolemy, who lived in Alexandria in Egypt during 150 AD, used a regular 360 polygon and followed the same method of Archimedes to approximate π. He actually obtained the number $3+8/60+30/60^2$, which if expressed as a decimal, comes as 3.1416666. This is accurate to the three decimal places.'

'Did anybody else use Archimedes methodology?'

'In the fifth century, Chinese mathematician Tsu Ch'ung used a variation of Archimedes' method to give the value of π as 355/113, which is actually in the range between 3.1415926 and 3.1415927. This value of π was correct up to 7 digits and mathematicians in Europe could not better this feat for approximately next thousand years.'

'What about our Indian mathematicians?'

'Aryabhata (476-550) made the approximation of π using regular polygon of 384 sides and he gave the value as 62832/2000, which is equal to 3.1416 and was correct up to four decimal places. Later Brahmagupta, who gave the operational concept of zero to the world, however gave the value of π as square root of 10, which is correct to only one decimal place.'

'No other Indian mathematicians tried to calculate π …'

'During the year 1400, another mathematical genius, Madhava, the mathematicians from Cochin, used a series to calculate π. He used the following series:

$$\pi/4 = 1 - 1/3 + 1/5 - ...$$

And from this series, he calculated the approximate value of π as 3.14159265359, which was correct up to 11 decimal places. Historically, this was a great achievement since his Europeans colleagues were still way behind this approximation during the same time.'

'Mathematics was also advanced in western Asia during that time as you have told me earlier how the concept of zero was taken by Arabic mathematicians (*see Dream 2047 March issue*). So I suppose that somebody from there must have tried to calculate π.'

'An Iranian mathematician, Jamshid al-Kashi used the principle of regular polygon of Archimedes and obtained the approximate value of π up to 14 decimal places.'

'What about the European mathematicians?'

'Another mathematician named Ludolph Van Ceulen used Archimedes methodology to calculate the value of π. In 1596, he succeeded in giving the approximate value of π up to 35 decimal places. There is an interesting story behind Ceulen's life. It is said that he was passionately engaged with

Srinivasa Ramanujan
(1887-1920)

the calculation of value of π in most of his life. So as a fitting tribute to him, the value of π up to 35 decimal places was engraved on his tombstone. In fact, in Germany, π was called '*die Ludolphsche Zahl*' or *Ludolphine number* for a long time.'

'So Archimedes method was continued till 16^{th} century.'

'During 17^{th} century, with the invention of calculus by Newton and Leibniz, the Archimedes' methodology to calculate the value of π was replaced with use of infinite series expansions. In the meantime, the concept on algebra and trigonometry were also developed to great extent. Moreover the concept of zero and decimal system of number made huge advancement in mathematics. Therefore, it was easy to interpret the problem of π taking help from all these branches of mathematics.'

'So π came out of the closet of geometry and embraces the arithmetic, algebra, trigonometry, calculus and all modern mathematics fields.'

'Yes you are right. For example, with the help of algebra, trigonometry and calculus, it can be proved that:

$$\tan^{-1} x = x - \left(\frac{x^3}{3}\right) + \left(\frac{x^5}{5}\right) - \left(\frac{x^7}{7}\right) + \left(\frac{x^9}{9}\right)....$$

This is well-known Gregory-Leibniz formula. In this formula, substituting x = 1, gives rise to the series, which was already used by our own mathematician Madhava long ago.

$$\tan^{-1} 1 = \frac{\pi}{4} = 1 - \left(\frac{1}{3}\right) + \left(\frac{1}{5}\right) - \left(\frac{1}{7}\right) + \left(\frac{1}{9}\right).$$

'That's really interesting!'

'However, one drawback of the above series is that it converges very slowly and so one would require to complete the series up to few hundreds terms only to compute the value of π accurately up to two decimal places.'

'When mathematicians were trying for getting the digits of π, I wonder whether Newton himself proposed any series or not.'

Archimedes
(287BC - 212BC)

'In 1665, Isaac Newton used the following series of arcsine:

$$\sin^{-1} x = x + \left(\frac{1\times x^3}{2\times 3}\right) + \left(\frac{1\times 3\times x^5}{2\times 4\times 5}\right) + \left(\frac{1\times 3\times 5\times x^7}{2\times 4\times 6\times 7}\right) + \ldots\ldots$$

It is known that: $\pi/6 = \sin^{-1}(1/2)$. So you can put the value of $x = ½$ to compute pi. Considering approximately 40 terms in the above expression, Newton computed the value of π, which was accurate up to 16 digits.'

'So one can put other values of x to get the value of π differently. For example, if I take x equals to $\sqrt{3}/2$, then I can consider the left hand side as $\pi / 3$.'

'You are absolutely right. In fact, that's what many mathematicians did in later years. In 1699, Abraham Sharp used Gregory-Leibniz series to compute the value of π. He considered the value of x in the Gregory-Leibniz series as $1/\sqrt{3}$. Now you know that: $\pi / 6 = \tan^{-1}(1/\sqrt{3})$. Therefore he was also able to get the value of pi which was up to 71 decimal places using approximately 300 terms of the series.'

'Were there more series like these?'

'In 1700s, Leonhard Euler provided some interesting series involving π. Some of these series involved expressions like $(\pi^2 / 6)$, $(\pi^4 / 90)$ and converged very rapidly. Later a faster and rapidly converged form of Gregory-Leibniz series was proposed by Machin in 1706. He used the following identity:

$$\pi / 4 = \tan^{-1}(1/5) - \tan^{-1}(1/239)$$

Using the similar principle of Gregory series for arctan(x), Machin approximated the π up to 100 decimal places. In 1874, William Shanks used the method of Machin and computed π up to 707 decimals, which however later found to be accurate only up to 527th place.'

'Uncle, let me make a little interruption. As you said earlier that the ratio of circumference of circle to the diameter was not known as π in earlier days, but some other names like *Ludolphine Number*. So when did we start associating the word π with this constant?'

'Well, you got it right. In the meantime, in 1706, the English mathematician William Jones assigned the value of 3.14159 to the 16th letter of Greek alphabet. He adopted π to represent this immensely significant value.'

'So from the beginning of 18th century π came into existence what we still call so.'

'Yes. After Jones' abbreviation of the value, Eular mentioned about this symbol in 1737 and soon it became a standard notation.'

'Let's go back to our main story. First it was Archimides method and then it was series. What's next?'

'Until the advent of computer technology in the mid 20th century, the computation of π was basically involved in calculation of the value in a series to the extent that is manually possible. Most of the calculation involved with series given by Gregory-Leibniz, Sharp and Machin. These series were not very efficient in computing the value of π. However, those series were very elegant in nature and useful in obtaining the approximation of π reasonably well to apply in practical circumstances. Moreover, those series gave many theoretical implications and research ideas, which are still being investigated by mathematicians around the world.'

'How the scheme of calculation of π was changed with the arrival of computer?'

'During the mid of 20th century, with development of computers and simultaneously some advanced algorithms for mathematical calculations, it was possible to obtain some efficient and accurate values of π and some other constants. However, until 1970s, all computer evaluations still used the classical formula like some variations of Machin's formula.'

'So, still there was no advanced algorithm to calculate π.'

'Well, it was not like that. Ramanujan discovered some new infinite series formula in 1910, but its importance was re-discovered around late 70s long after his death. One of his elegant formulas was like this:

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103 + 26390k)}{(k!)^4 396^{4k}}$$

'I always amazed hearing stories about our own mathematician genius Ramanujan. He was a mathematician of extraordinary calibre!'

'You are right. If he would not have died at younger age, he must have contributed to the world of mathematics a lot more.'

'So Ramanujan's series advanced the computation of digits in π.'

'Yes, with each addition of term in Ramanujan's series could give approximately additional eight digits to pi. During the year 1985, 17 million digits of π were accurately computed by Gosper using this formula. So it also proved the validity of Ramanujan's formula. In 1994, David and Gregory Chudnovsky brothers of Columbia University computed over four billion digits of π in a supercomputer, using an algorithm, which was also similar in essence to the formula given by Ramanujan.'

'Could anybody improve Ramanujan's formula?'

'In 1976, Eugene Salamin and Richard Brent independently discovered a new algorithm for π, which was based on arithmetic-geometric mean iteration or in short, AGM iteration. Their algorithm was faster than Ramanujan and with 25 iterations, 45 million digits of π can be calculated accurately.'

'That's huge number of digits.'

'Well, there were still many to come. In 1985, Jonathan Borwein and Peter Borwein discovered some additional algorithms. Using their algorithm along with Salamin-Brent scheme, Yasumasa Kanada of the University of Tokyo computed 6.4 billion decimal digits of π on a Hitachi supercomputer in 1999.'

'That must be a world record for calculation of digits of π.'

'To be precise, that is history now! In December 2002, Kanada and his group broke their own world record and calculated value of π for 1,2411 trillion places.'

'Wow! It's beyond my imagination! How big is it?'

'You can judge this gigantic feat by the fact that it will take almost 40,000 years to recite all digits. Professor Kamada used a Hitachi supercomputer that was capable of performing two trillion calculations per second and it took 400 hours to compute the calculation of those 1.2 trillion digits.'

'Was there any new algorithm after Borwein algorithm?'

'In 1990, another algorithm, called Rabinowitz-Wagon spigot algorithm, was proposed for computation of π. The characteristic feature of the algorithm was that previously generated digits could be used in generation of next successive digits.'

'All these algorithm may be only applicable in a high speed computer like supercomputer as that of Kanada.'

'Yes, all these algorithms are computationally very exhaustive. Most of these algorithms require the computation of previous digits to get the next digit. For examples, to get the n^{th} digit in π, computer should first compute all previous (n-1) digits.'

'Is their any algorithm which can calculate n^{th} digit without calculating (n-1) digit?'

'Mathematicians have found that this may be possible for binary (base 2) and hexadecimal (base 16) digits of π. In 1996, D. Bailey, P. Borewein and C. Plouffe discovered a novel scheme of computing individual hexadecimal digits of π. The uniqueness of their scheme is that it can produce modest length of binary or hexadecimal bits from any arbitrary position using no prior bits and it can be implemented in any modern computer without any multi-precision software or higher memory. More recently in 1997, C. Plouffe discovered another new algorithm to compute the *n*th digit of π in any base.'

'So one can calculate any digit of π in any position using this algorithm.'

'Using Bailey's algorithm, Colin Percival, a 17-year student from Simon Fraser University, calculated five trillionth and ten trillionth hexadecimal digit of π. In the year 2000, he found that the quadrillionth binary digit of π is zero. And more recently, to add another feather in their cap, Kanada group also finished computing 1,030,700,000,000 hexadecimal digits of π.'

'It's amazing that mathematicians from ancient time to modern age were engaged with the calculation of digits in π. But still I'm wondering about one thing! I agree that understanding digits of π is important, but all these trillion of digits…'

'Well, π was always a mystery to mathematician and so they might have tried to get to the bottom of it. A value of π for just 37 places is sufficient to for mathematicians to calculate the radius of the Milky Way galaxy with a margin of error less than the size of a hydrogen atom. So it is really interesting to see that mathematicians all over the world are so fascinated and engaged to get trillion digits of π when for the purpose of the most accurate measurement, it does not require even first hundred digits!'

'Yes, I have also the similar thoughts.'

'One reason is that calculation of digits of π is an excellent way to judge the power and integrity of our modern days computer hardwares and softwares. If two computers compute the billionth digit of π accurately, then we can assume that these two computers are reliable for doing millions of other calculations flawlessly. One can detect the problems in hardware after obtaining the results of π digits. The similar kind of problem was once detected in Cray-2 supercomputers in 1986.'

'Well, that makes sense. It's indeed a great exercise to test the ability of minds of supercomputer!'

'Moreover, the challenge of computing π has also stimulated researches in many advanced areas of science and engineering. The challenge has led to many new discoveries and many new algorithms in the field of mathematics. So these were the added benefits that we obtained from this mysterious constant. There are also academic interests to find any statistical abnormalities or irregularities in π that could suggest that π is not a normal number.'

As we were talking, the big wall clock in our drawing room told us that it is 10 o' clock in the night. Uncle stood up giving a look at the clock.

'My dear Googol, I think that we should stop now. You have to go to school tomorrow. So run to bed and have a good night sleep!'

I took uncle's word and started preparation to retire for the night. But by then, the magic of pi already mesmerised my mind completely.

A Glance at the Golden Ratio

Last week, I went to watch a Charlie Chaplin movie titled 'The Gold Rush' in a retrospective film festival. It is a characteristic Chaplin comedy interwoven with a mixture of different human emotions. I was talking about the film with my uncle that evening. He told me that the theme of the film was indeed based on a true incidence. The Klondike Gold Rush (also called the Yukon or Alaska Gold Rush), was a migration of an estimated 100,000 prospectors between 1896 and 1899 to the Klondike region of the Yukon in north-western Canada where a large deposit of gold was discovered. It was an extremely difficult journey in a very rough terrain and cold climate, and only 30,000 could reach at the destination while only 4000 were succeeded to find gold.

'Gold always remained a precious metal for all civilisations, and the history down the years was written around the gold,' I commented.

'Yes, the touch of gold can be found everywhere - from the history to the modern day finance.'

'An example, uncle...'

'Well, here is a clue for you. Can you find the connection between these words: age, mean, rule, share, goal, ratio?'

'I guess that the connection is something associated with gold.'

'You guessed it correctly. The connection is the adjective form of the noun 'gold'.

'The adjective form of gold is golden.'

'You got it right. You can use the adjective *golden* for all these words'

'Let me give it a try. The first one is: 'golden age'. I know that it's often quoted in the historical incidence. For example: the golden age or the golden era of the Moghul dynasty...'

'Good Googol. In History, 'golden age' refers to the period when an activity, art, skill etc. was at its peak or the period that encompasses peace, prosperity and happiness of people.'

'The next ones are 'golden mean' and 'golden rule'. I'm not much sure about these!'

'Philosophically, the terms 'golden mean' and 'golden rule' have special meanings. In philosophy, especially that of Aristotle, the 'golden mean' is the desirable middle path between two extremes, one of excess and the other of deficiency. For example, the 'courage', a virtue, if taken to excess would manifest as recklessness, and if deficient as cowardice. The 'golden rule' or 'golden ethic' has also philosophical connection. It means the ethical code or morality that essentially states one should treat others as one would like others to treat oneself.'

'The word 'golden share' must be associated with the financial world.'

'You are right. Mostly in Britain, a share in a company that gives control of at least 51% of the voting rights, especially when held by Government is termed as a 'golden share'. The precious metal is of course very important for the economy of any country and it is characterised by 'gold reserve' which is the quantity of gold held by a central bank of a country.'

'I know about the next one. The 'golden goal' is the first goal scored during the extra time of a football match. The golden goal ends the match and gives victory to the scoring side.'

'Fantastic.'

'And the last one is 'golden ratio'. I presume that this must have some connection with mathematics.'

'You guessed it right Googol. The golden ratio is one of the most simple, elegant and beautiful ratios of the mathematical world.'

'I don't know anything about the golden ratio. Please uncle, tell me more about the golden ratio.'

'Well, before that let me see how you could make the golden connection here in this number series. Could you please tell me the connection between these numbers: 1, 1.618, 50 and 79?

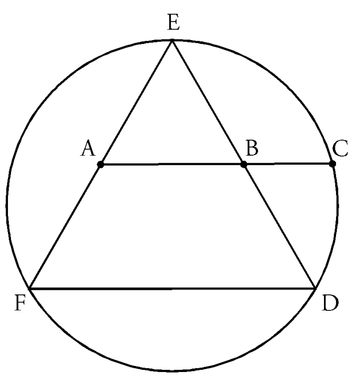
'Hmm, let me try again uncle. I can see the connection in two of these. If a person secures the *first* position in a race or competition, he or she is awarded the *gold* medal. The *fiftieth* anniversary of a significant event is called *golden* jubilee or anniversary.'

'Very good, Googol. And what is the chemical symbol of gold?'

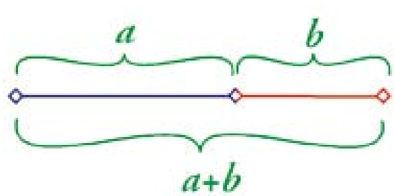
'The chemical symbol of gold is Au. Yes I got it now. The *atomic number* of *gold* is 79.'

'And finally, the number 1.618 represents the *golden ratio*.'

'That's very interesting. What's special about this number?'



$$\frac{|AB|}{|BC|} = \frac{|AC|}{|AB|} = \Phi$$



$$\frac{a+b}{a} = \frac{a}{b} = \Phi$$

"What is the Golden Ratio?"

"Golden ratio is defined as a line segment divided into two unequal parts, such that the ratio of the longer portion to the shorter portion is same as the ratio of the whole length to the longer portion. It is believed that this ratio is found throughout nature and is an integral part of art, architecture, music, philosophy, science, and mathematics."

"What is the value of the Golden Ratio?"

"The precise value of the golden ratio is a never ending and never repeating number 1.6180339887…., and such never ending numbers have intrigued humans since antiquity. The Golden Ratio is denoted by a symbol Φ. A variant of golden ratio is called the golden rectangle "

"What is golden rectangle? "

"A rectangle, whose side lengths are in the golden ratio, or approximately 1:1.618. A distinctive feature of the golden rectangle is that when a square section is removed, the remainder is another golden rectangle; that is, with the same proportions as the first. Square removal can be repeated infinitely, in which case corresponding corners of the squares form an infinite sequence of points on the golden spiral, the unique logarithmic spiral with this property. Many artists and architects have been fascinated by the presumption that the golden rectangle is considered aesthetically pleasing."

"It is interesting to see mathematics in arts"

"Yes Googol. Many artists and architects have proportioned their works to approximate the golden ratio—especially in the form of the golden rectangle, in which the ratio of the longer side to the shorter is the golden ratio—believing this proportion to be aesthetically pleasing. Mathematicians have studied the golden ratio because of its unique and interesting properties."

"Please tell me how the golden ratio was discovered"

"Ancient Greek mathematicians first studied the golden ratio because of its frequent appearance in geometry. The division of a line into the golden section is important in the geometry of regular pentagons and pentagrams. Euclid defined a proportion derived from a simple division of a line into what he called its 'extreme and mean ratio' uncle replied.

How to construct a golden rectangle

Construct a simple square of unit length (say AB = 1 inch, in figure 1). Draw a line from the midpoint (E) of one side (AB) of the square to an opposite corner (C). Use this line (EC) as the radius to draw an arc (EC = EG) as shown in figure 1. Complete the rectangle AGFD.

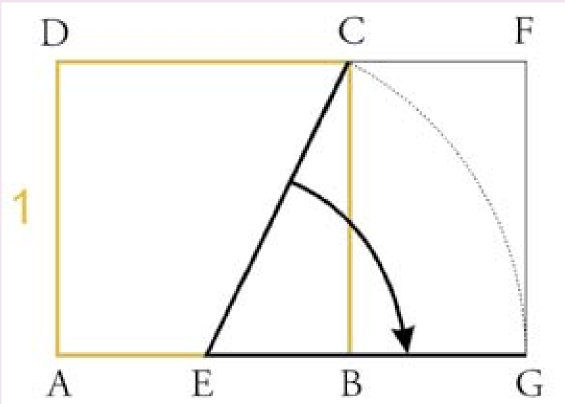


Figure 1

Now, BC = 1, EB = ½

Using Pythagoras theorem, $EC^2 = EB^2 + BC^2 = (1/2)^2 + (1)^2$ $= ¼ + 1 = 5/4$

Therefore EC = √ 5/2 = EG

AG = ½ + √ 5/2 = (1 + √ 5)/2 = 1.618

Ratio of the sides = AD: AG = 1 : 1.618

The rectangle AGFD is golden rectangle. From this rectangle, if the square ABCD is removed, the remaining rectangle BGFC becomes another golden rectangle.

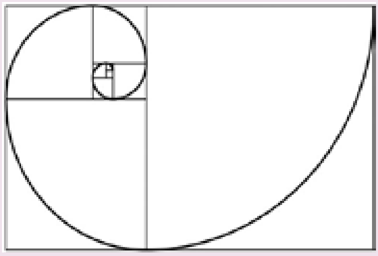
One startling feature of the golden ratio is that we produce its square by simply adding the number 1; i.e., $\Phi^2 = \Phi + 1$.

"Who discovered it?"

"Euclid's *Elements* (300 B.C.) provides the first known written definition of the golden ratio. In Euclid's words:

Golden ratio in nature

The following two figures show the construction of a golden spiral and its nearest match in nature (mollusc shells).



In case of the daisy flower, the florets that make up this pattern (here represented by arcs) grow at the meeting points of two sets of spirals, which move in opposite directions, one clockwise, the other counter clockwise. If we connect the consecutive meeting points of these two sets of opposite lines, we can see the daisy's growth spirals. These spirals are logarithmic and also equiangular, since the angle they describe with the radii remain always the same.

Daisy flower

'A straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the lesser.''

Golden ratio in architecture

Height of pyramid = h

If $h^2 = a \times b$ then $a/b = \Phi$

Using Pythagoras theorem,

$a^2 = h^2 + b^2$

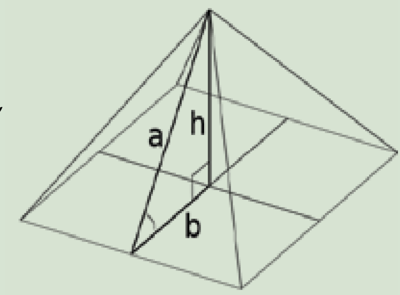
=> $a^2 - b^2 = h^2 = a \times b$

=> $a^2 = a \times b + b^2$

=> $a^2/b^2 = a/b + 1$

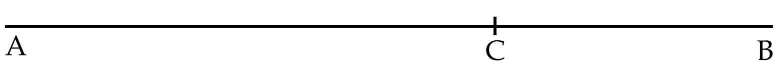
If $a/b = \Phi$, then

$\Phi^2 = \Phi + 1$, which is the feature of golden ratio.



Both Egyptian pyramids and those mathematical regular square pyramids that resemble them can be analysed with respect to the golden ratio. A pyramid in which the apothem (slant height along the bisector of a face) is equal to φ times the semi-base (half the base width) is called a golden pyramid.

Some artists and architects believe the golden ratio makes the most pleasing and beautiful shape. Many buildings and works of art have the golden ratio in them.



If the ratio of the length AC to that of CB is the same as the ratio of AB to AC, then the line has been cut in extreme and mean ratio, or in a golden ratio.

Evidence exists that the ratio may have been known to the ancient Egyptians (1650 BC). Egyptians referred it as the "sacred ratio". The ratio of the altitude of a face of the Great Pyramid at Gizeh to half the length of the base is approximately 1.618. Through the ages other names have been attached to this wonderful ratio including golden mean, golden number, and divine proportion. '

‘Is the golden ratio irrational number?’ I wanted to know.

‘Yes. However, during the fifth century B.C. there was no concept of irrational number. Hence a number that is neither a whole number nor even a ratio of two whole numbers (like fractions1/2, 2/3, 3/4) absolutely shocked the mathematicians. That is why the Golden ratio did not get immediate acceptance with mathematicians. The Pythagorean worldview was based on extreme admiration for the numbers – the intrinsic properties of whole number or their ratios – and their presumed role in the cosmos. The realisation that there exist numbers, like the golden ratio, that go on forever without displaying any repetition or pattern caused a true philosophical crisis.’

‘Please tell me what happened after that.’

‘The modern history of the golden ratio starts with Luca Pacioli's *De Divina Proportione* in 1509, which captured the imagination of artists, architects, scientists. Italian mathematician Bartolomeo de Pacioli (also known as Luca Pacioli) wrote a book *De Divina Proportione* (About divine proportions). The subject was mathematical and artistic proportions and the book was illustrated by Leonardo da Vinci. The first part of the book describes the golden ratio from a mathematical point of view and also studies polygons.’

‘It is interesting that Leonardo da Vinci’s name is also associated with the Golden ratio. Did he use the Golden ratio in his paintings?’

‘Some scholars speculate that Leonardo da Vinci incorporated the golden ratio in his paintings. However, it is not supported by Leonardo’s own writings. Leonardo da Vinci’s illustrations of polyhedra in the book *De Divina Proportione* were based on the golden ratio and he was of the view that some bodily proportions exhibit the golden ratio.’

‘It is interesting to know that the Golden ratio is important not only in mathematical world, but also in arts.’

‘Indeed it is. Some of the greatest mathematical minds of

Golden ratio and the Fibonacci series

Fibonacci sequence is a recursive series of numbers where the following number is equal to the sum of the previous two. The sequence goes like, 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, …….. and so on. The Fibonacci series is named after Italian mathematician Leonardo of Pisa (1170 – 1250) (more commonly known as Fibonacci.

There is a special relationship between the golden ratio and the Fibonacci series. Ratio of any two successive numbers in Fibonacci series is close to the golden ratio (1.618025....).

A	B	B/A
2	3	1.5
3	5	1.666666666...
5	8	1.6
8	13	1.625
13	21	1.615384615...
... ...	...	
144	233	1.618055556...
233	377	1.618025751...

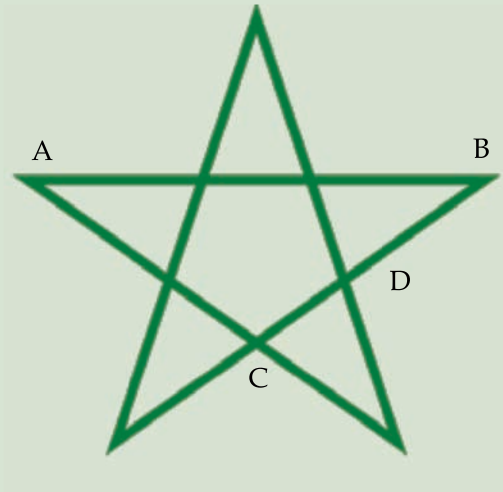
- $\Phi^2=1\Phi+1$
- $\Phi^3=2\Phi+1$
- $\Phi^4=3\Phi+2$
- $\Phi^5=5\Phi+3$
- $\Phi^6=8\Phi+5$
- $\Phi^7=13\Phi+8$
- . . .
- $\Phi^n=F(n)\Phi+F(n-1)$

F(n) is the n^{th} Fibonacci number. Notice the Fibonacci Numbers on the right side of each equal sign (numbers in red and blue, separately form Fibonacci series).

Like golden ratio, there are amazing connections between Fibonacci numbers and natural forms (number of spirals in a pine cone, sunflower seed arrangement). There are boundless applications of Fibonacci series in geometry, number theory, probability, and algebra, to name but a few.

All of these are astounding evidence of the deep mathematical basis of the natural world. The golden ratio and the Fibonacci series is evidence of the beauty of mathematics. The amazing phenomenon permeates just about everything - both in and outside of the world of mathematics.

Pentagram



The golden ratio plays an important role in constructing regular pentagrams.
AB/AC = BC/BD = BD/DC = Φ

all ages, from Pythagoras to Euclid in ancient Greece, through the medieval Italian mathematician Leonardo of Pisa and the Renaissance astronomer Johannes Kepler, to present day scientists, have spent endless hours over this simple ratio and its properties. But the fascination with the golden ratio is not just confined to mathematicians, biologists, artists, musicians, historians, and architects; psychologists have pondered and debated the basis of its ubiquity and appeal. In fact, it is probably fair to say that the golden ratio has inspired thinkers of all disciplines like no other number in the history of mathematics.'

'Please tale me some applications of the Golden ratio,'

'In 2010, the journal *Science* reported that the golden ratio is present at the atomic scale in the magnetic resonance of spins in cobalt niobate crystals. Researchers have for the first time observed a nanoscale symmetry hidden in solid state matter. They have measured the signatures of a symmetry showing the same attributes as the golden ratio. The observed resonant states in cobalt niobate are a dramatic laboratory illustration of the way in which mathematical theories developed for particle physics may find application in nanoscale science and ultimately in future technology.'

'Uncle, I think that Charlie Chaplin is the Golden Ratio of the film world.'

'And why do you think so, Googol?'

'The Golden Number is an example of the beauty and elegance of the complex mathematical world while Charlie Chaplin represents the slapstick and comedy face of the mundane, monotonous and serious human life.'

'That's indeed a reasonable comparison.'

The Enigmatic 'e'

My uncle and I were on our daily evening stroll when we spotted a branch of tree that blocked our path.

'Googol, please help me move this branch to the side of the path – it will be less visible in the night and may cause problem to other walkers,' uncle said.

'It must be due to the strong wind that blew yesterday evening,' I tried to reason as I was helping him to clear the path.

'I think so. Well Googol, here is a riddle for you. How does a mathematician describe a large branch of tree that has fallen off due to natural cause?' uncle asked.

'A mathematician will find mathematics in the fallen branches of tree!' I was a bit amused.

'Natural log,' pat came the reply from uncle.

'Oh, I got it now. The 'log' is also the short form of 'logarithm' and the 'natural logarithm' of a number is obtained by taking the base as *e*,' I replied.

'You are right, the pun was intended,' uncle smiled.

'Uncle, I don't know much about the number '*e*' – could you please elaborate it?' I was eagerly waiting for a breeze of mathematical ideas from my mathematician uncle.

'The number *e* is one of the most fascinating mathematical constants. It appears in natural logarithm, calculus and in many mathematical equations. Decimal representation of *e* never ends and never repeats,' uncle said.

'I knew π is a very interesting number and it's also very useful. So *e* is also an interesting constant. Historically, were they discovered around the same time?' I asked.

'Well, history of π goes back to ancient times. The Egyptians mentioned about something similar to π in their writings on papyrus scroll as early as 1650 BC. Moreover, its concept can be grasped easily as the ratio of the circumference of a circle to its diameter. However, it was not the case with the number *e*. Not only the concept of *e* came much later, around the year 1700, its history is closely associated with calculus, the subject traditionally regarded as higher mathematics. To mathematicians, however, *e* is equally important as π.'

The person who discovered *e* must have been a genius!'

'Indeed he was. But it's not easy to decide who should be given the credit as the discoverer of *e*'.

'Does it mean several mathematicians were involved in discovering the number *e*?'

'It appears the number *e* was known to mathematicians long before the invention of the logarithm and calculus. It first appeared in connection with the formula for calculating compound interest'.

'How could the formula for calculating compound interest be related to *e*?

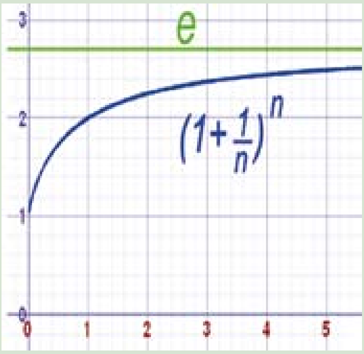
'With an annual interest rate of 100%, how much will you get from a bank after one year if your principal is Rs 1?' uncle asked.

'Very simple. I will get Rs 2 after one year.'

'Now assume that the bank is calculating the compound interest and it is calculated half-yearly. Can you tell me how much will you get at the end of one year?'

'Hmm, compounding the interest will give a different answer. It is not so simple now − please help me,' I confessed.

'Yes, compound interest is the interest on the principal amount plus the interest accrued on the principal amount over a time period. One of the formulae for calculating the compound interest can be expressed mathematically as: $S = P(1 + r/n)^n$, where S is the total amount you will receive, P is the principal, r is the rate of interest expressed as decimal (and not as percentage), n is the number of times the interest is



n	$(1 + 1/n)^n$
1	2.00000
2	2.25000
5	2.48832
10	2.59374
100	2.70481
1,000	2.71692
10,000	2.71815
100,000	2.71827

The value of $(1 + 1/n)^n$ approaches *e* as *n* gets bigger and bigger

compounded per year. With P = 1, r = 1 (i.e. 100%), the formula becomes $S = (1 + 1/n)^n$. For half yearly compound interest calculation, you calculate compound interest two times a year and hence put n = 2'.

'Thanks for your help. I will get: $S = (1 + ½)^2 = (1.5)^2 = 2.25$,' I replied.

'Now, if you increase 'n' to 4, i.e. if interest is calculated quarterly, $S = (1 + ¼)^4 = 2.44$. If n is increased further, say n = 12, $S = (1 + 1/12)^{12} = 2.613$. If you keep increasing n, S will approach a limit, about 2.71828. This observation was made during 16 century'.

'That is amazing! Is this the value of *e*?'

'Yes, you got it right – this is the approximate value of *e* for five decimal points.'

'I will memorise this, 2.71828...,' I mumbled.

'Or, there is an easy way to remember the value of *e* to some digits. Remember the curious pattern that after the 2.7, the number '1828' appears twice. This gives: 2.7 1828 1828,' uncle said.

'That's nice; it's easy to remember then,'

'And then follows the numbers in degrees of the angles of a right-angled isosceles (two equal angles) triangle. This gives us the values: 45, 90, 45. So if you put all of these together, you get the value of *e* to a considerable length: 2.7 1828 1828 45 90 45,'

'How many digits of *e* are known today?'

'Interestingly, the number of known digits of *e* has increased dramatically during the last few decades. This is due both to higher performance of computers and to algorithmic improvements. In July 2010, Shigeru Kondo and Alexander J. Yee computed the value of *e* up to 1,000,000,000,000 digits.'

'That's very interesting. So who discovered this unique mathematical phenomenon for the first time and hence the value of *e*?'

'The discovery, most likely was an experimental observation rather than the result of rigorous mathematical deduction. Hence it is difficult to give the credit to any individual. The result is fascinating because inadvertently the concept of limit was introduced in this financial calculation. Note, $S = (1 + 1/n)^n$ with n approaching infinity, S approaches *e*.'

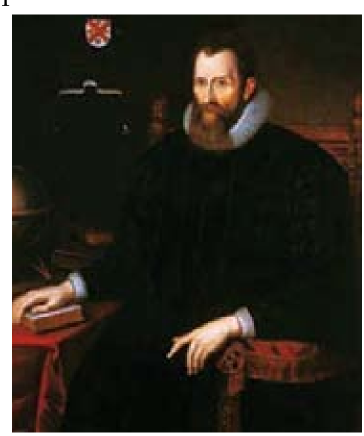
'When did mathematicians knew about *e*?' I asked uncle.

'The familiar role of *e* as the natural base of logarithms came much later. Scottish theologian and mathematician John Napier, while trying to simplify multiplication, invented a model which transforms multiplication into addition and hence came up with the idea of logarithm. He created first table of logarithms in 1614. The model is almost equivalent to what we know as logarithm today,

$y = \log_b x$ if $b^Y = x$ '

'Give me an example.'

'For example, if $b = 2$ and $Y = 4$, then: $b^Y = 2^4 = 2 \times 2 \times 2 \times = 16 = X$. If you now take the logarithm of 16 with base 2, then it gives 4 which is Y. So the logarithm helps in simplifying



John Napier
(1550 – 1617)

the concept of multiplication. In fact, for any value of the base, log(M×N) equals to log(M) plus log(N)'

'I know that the logarithm table is used for mathematical calculations'.

'Not only for computational mathematics; logarithmic functions are central to almost every branch of pure and applied mathematics.'

'So the application of logarithmic function is not restricted to the field of mathematics only.'

'Yes, the logarithmic functions are essential in a host of applications, ranging from physics and chemistry to biology, physiology, art and music.'

'But how is *e* associated with logarithm?'

'Napier's work was translated in 1618 where, in an appendix, there is equivalent statement that $\log_e 10 = 2.302$. This seems to be the first explicit recognition of the role of the number *e* in mathematics.'

'So Napier invented the logarithm with base *e*?'

'Although Napier's definition did not use bases or algebraic equations, he did use a number close to 1/*e* as the base. Algebra was not advanced enough in Napier's time to allow such a definition. Logarithmic tables were constructed; even tables very close to natural logarithmic tables, but the base, '*e*' did not make a direct appearance'

'So Napier did not mention anything about *e*'.

'You may say that. However, Napier unknowingly came very close to discovering the number *e*, which, a century later, was recognised as the natural base of logarithm'.

'I am eager to know when did *e* get its recognition as a mathematical constant.'

'German mathematician and philosopher Gottfried Leibniz, in his work on calculus, identified a constant of value 2.718 and labelled it *b*.'

'But that is the value of *e*!'

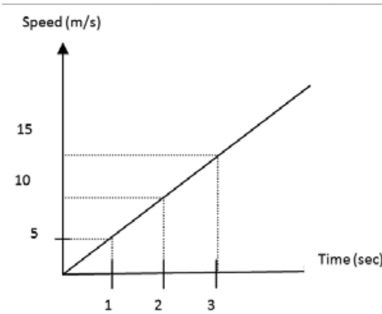
Leonhard Euler
(1707 – 1783)

'Yes. But It was Leonhard Euler who gave the constant its letter designation, '*e*', and discovered many of its remarkable properties. Euler's discoveries cast new light on the previous work, bringing out the relevance of *e* to a host of results and applications.'

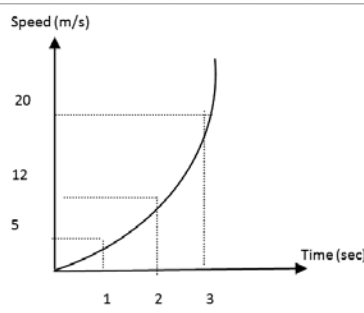
'I have heard about exponential growth. Does it have anything to do with *e*?'

'Exponential growth signifies nonlinear increase. However mathematical equations representing exponential growth is not necessarily be a function of *e*.'

'Please elaborate'



Linear growth

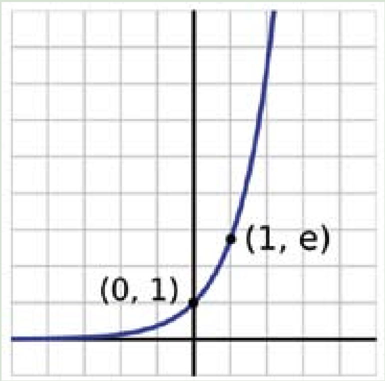


Non-linear growth

'Say, a car starts from rest. Its speed after 1 second is 5m/s, after 2 sec 10m/s, after three second 15 m/s and so on. Now if you want to plot a graph with speed in y axis and time in x axis, you will get a straight line with a positive constant slope with x axis. This is a linearly increasing function. However, if speed of the car after 2 second is 12 m/s, after 3 second 20 m/s, then it is an example of exponential growth and the graph will be a curve, not a straight line. In this case the slope is different at different intervals.'

'Can *e* represent this exponential growth?

'Any function of the form $y = b^x$, where the base b is any positive real number and x is a real or a complex number is called exponential function. If x is positive, the value of y will increase exponentially. If x is negative, the value of y will decrease exponentially, known as exponential decay.'



The natural exponential function y = ex. The graph of $y = e^x$ is upward-sloping, and increases faster as *x* increases. The graph always lies above the *x*-axis but can get very close to it for negative *x*

'I thought *e* must be related to exponential!'

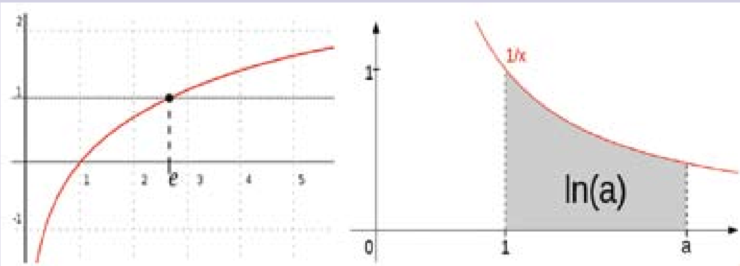
'You are partially right. The most common base is the number *e*. The function $y = e^x$ is called exponential function. When the exponent in this function increases by 1, the value of the function increases by a factor of *e* .The beauty of this function is that derivative of e^x is e^x. Hence the function is used to model a relationship in which a constant change in the independent variable gives the same proportional change (i.e. percentage increase or decrease) in the dependent variable.'

'Can exponential function be related with natural logarithm?' I enquired.

'Very good question. If we take natural logarithm of the exponential function $y = e^x$ we get $\log_e y = \log_e e^x = x \log_e e = x$. This is a equation with natural logarithm and the curve becomes straight line if plotted in a logarithmic scale.'

'This is amazing!'

Natural Logarithm



The natural logarithm is the logarithm to the base *e*. The natural logarithm is generally written as ln(x) or $\log_e x$.

The natural logarithm of a number x is the power to which e would have to be raised to equal x. For example, ln(7.389...) is 2, because e^2=7.389.... The natural log of *e* itself [ln(*e*)] is 1 because $e^1 = e$, while the natural logarithm of 1 [ln(1)] is 0, since $e^0 = 1$.

The natural logarithm can be defined for any positive real number 'a' as the area under the curve y = 1/x from 1 to 'a'. The simplicity of this definition, which is matched in many other formulas involving the natural logarithm, leads to the term 'natural'.

Alternative characterisations of *e*

The number *e* is the sum of the infinite series

where *n*! is the factorial of *n*. For example, 5! = 5×4×3×2×1 = 120.

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

From the infinite series it is clear that decimal representation of *e* can never end and can never repeat!

The numerical value of *e* truncated to 50 decimal places is

2.71828182845904523536028747135266249775724709369995...

'Indeed it is. The mathematical constant *e* transforms a complex looking expression to a very simple form. Thanks to Napier for his ingenuity and effort in making the logarithmic table. The logarithm and its close associate e enable scientists to do complex mathematical modelling that otherwise would have been impossible.'

'Thank you for telling me about the astonishing features and the importance of *e*. Next time I come across any equation involving logarithm and *e*, I will try to visualise it in the light of whatever you told me. Please tell me more about the properties of *e* according to the number theorem.'

'Googol, if we dwell more on mathematics now, we will inevitably miss the beauty of nature that is just unfolding in front of us. So let's keep our discussion on more enchanting facts of Euler's number for tomorrow.'

Uncle was right. While walking and listening to uncle, I did not realise that the setting Sun was glowing in the horizon and the rain-washed sky was looking like a nature's easel splattered with dazzling colours. After an enchanting encounter with Euler's number, it's time to enjoy the nature!

Niceties of Numbers

This summer we decided to spend our holidays in our ancestral village. It was indeed a very nice experience. Far away from the hustle and bustle of city, the tranquillity and freshness of village life captured my mind. My mathematician uncle and I used to take a long walk across the muddy road of the village in early morning. The gentle cool breeze, the mesmerising chirping of birds, the intoxicating smell of wet soil, the eye-catching greenery of crops and the blue sky in the horizon all added together to give an invigorating and enthralling experience.

One such morning, my uncle and I were walking down the village path. I could hear that my uncle was humming a song. I tried to guess the song, but could not get it. I could not hide my curiosity about the unfamiliar song that he was humming.

'What are you singing, uncle?' I asked.

'Let's put it this way: it's a singular mathematical song,' my uncle replied.

It was expected that my mathematician uncle will think about mathematical problems. But it was a little surprising to me that he would sing a mathematical song. I was interested to know what the song could be.

'I thought so – it should be something mathematical,' I quipped.

'My dear Googol, you didn't get it. What's the word that connects the expression *singular mathematical song*?', my uncle asked with a smile.

I was little puzzled. No doubt, it was his very characteristic cryptic clue.

Sir, I bear a rhyme excelling
3 1 4 1 5 9
In mystic force, and magic spelling
2 6 5 3 5 8
Celestial sprites elucidate
9 7 9
All my own striving can't relate
3 2 3 8 4 6
Or locate they who can cogitate
2 6 4 3 3 8
And so finally terminate. Finis.
3 2 7 9 5

Thirty-one decimal places of π:
π = 3.14159 265358 979 323846 264338 32795

'Number,' uncle gave a short reply.

'I can see the connection of number with mathematics, and other connections also look plausible,' I was trying to put together the clues in the puzzle.

'Yes, you're getting there. A *number* is a grammatical classification of words that consists typically of singular and plural. The number system, as you know, is an important component of mathematics. And you might have heard that sometime a song, dance or other musical item is referred to as a number.'

'Yes, I got it now. So what was that mathematical song?'

'Well, I just made a song out of a nice mnemonic poem on pi that gives thirty-one decimal places of pi.'

'Please tell me the poem.'

'Sir, I bear a rhyme excelling / In mystic force, and magic spelling / Celestial sprites elucidate/ All my own striving can't relate / Or locate they who can cogitate / And so finally terminate. Finis.'

'Hmm, that's fascinating. The number of letters in each word is giving the value of pi: 3.141592, etc. I will try to memorise it later. But uncle, please tell me more about the number system. The other day you mentioned this to me.'

'The number system dates back to very early age of mathematical thinking. Greek philosopher and mathematician Pythagoras and his followers believed that numbers are the prime cause behind everything in the world, from the musical harmony to the motion of planets and the formation of the Universe. By 'number' they meant natural numbers', uncle replied.

'But how can numbers be the prime cause behind everything?' I wanted to know.

'Well, that's what Pythagoreans believed. However, they were not far from reality. Numbers may not be the cause, but they are required to explain everything. Numbers are the soul of mathematics and without mathematics there will be no understanding of the world around us.'

'Uncle, you said natural numbers – what are they?'

'The natural numbers are the ordinary whole numbers used for counting; for example, five fingers, two apples, etc.'

'Is zero a natural number?'

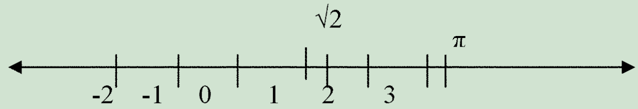
'Tricky question! In fact, there is no universal agreement whether to include zero in the natural number set. Some mathematician define natural numbers as the set of only positive integers like {1, 2, 3....}, while other mathematicians say it is the set of non-negative integers {0, 1, 2, ...}'.

Integers can be thought of as discrete, equally spaced points on an infinitely long number line.

'What is an integer?'

'In Latin, 'integer' means 'untouched', therefore one can say a whole number. The word 'entire' comes from integer. Integers are a subset of the real numbers – they are numbers that can be written without a fractional or decimal component, and fall within the set {...,−3, −2, −1, 0, 1, 2,3, ...}. For example, 7, 39, and −431 are integers; 9.75, −5½, and $\sqrt{2}$ are not integers.

'Uncle, you said integers are subset of real numbers, does it mean real numbers include numbers that are not whole



Real number line with rational and irrational numbers

numbers?'

'Yes, real numbers include not only integers, but their ratios, called fractions; for example, 2/9, 7/5, − 6/5, etc. Both integers and their fractions are called rational numbers. Real numbers also include irrational numbers, numbers which cannot be represented as the ratio of two integers. '

'Oh! There are so many types of numbers. I am lost in numbers!'

'Let me explain. The whole world of numbers may be divided into two types, 'real numbers' and 'imaginary numbers.' The real numbers include all the rational numbers, such as the integer −5 and the fraction 4/3, and all the irrational numbers such as $\sqrt{2}$, π, e, etc. Real numbers can be thought of as points on an infinitely long line called the number line or real line. Integers are equally spaced on real line.'

'What is a rational number?'

'As I mentioned before, a rational number is a number that can be expressed as the quotient or fraction a/b of two integers, with the denominator b not equal to zero', uncle explained.

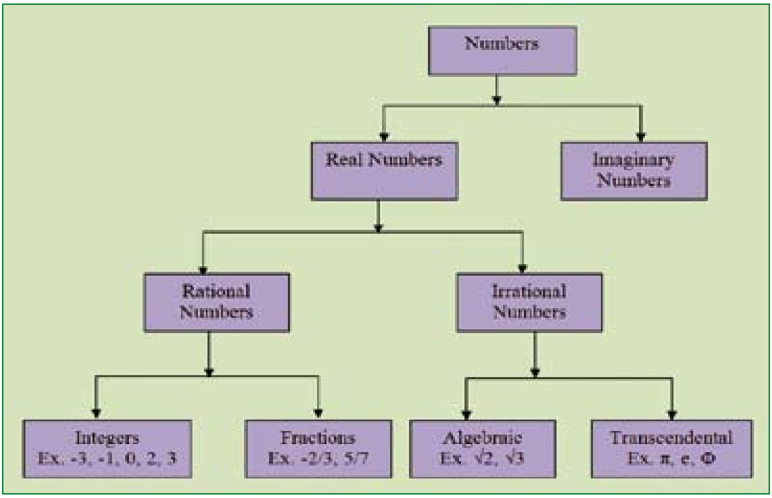
'Are all the integers rational numbers?' I wanted to know.

'Yes, you are right. For example, the number 4 is an integer as well as rational as it can be written as 4/1', uncle replied.

'You said integers are equally spaced; does it mean

π^e, π^π, e^e, e^π

Are these numbers transcendental or algebraic?



fractions are not equally spaced?'

'Fractions form a *dense* set of numbers. Between any two fractions, no matter how close we go, we can always find another number. Take the fractions 1/1001 and 1/1000 as an example. These fractions are certainly close. Yet we can easily find a fraction that lies between them, for example 2/2001. We can repeat the process and find a fraction between 2/2001 and 1/1000, for example 4/4001. Not only there is room for another fraction between two given fractions, there are infinitely many new fractions. Consequently we can express the outcome of any measurement in terms of rational numbers alone.'

'That means entire number line is populated by rational numbers', I said.

'That seems to be natural conclusion. However it is not true.'

'You mean to say that the number line is not continuous with rational numbers?'

'That precisely the point – within rational numbers there are many irrational numbers.'

'Yes, you have mentioned it earlier. What is an irrational number?'

Most real numbers are irrational, and among irrational numbers, most are transcendental

'An irrational number is a real number that cannot be written as a simple fraction. It cannot be represented as terminating or repeating decimals. For example, square root of 2 ($\sqrt{2}$) is an irrational number as it cannot be represented as a/b form, where a and b both are integers and b is non-zero.'

'I did not get that – please give me an example.'

'The number 1.4 can be expressed as 7/5. This signifies 1.4 is a rational number. However number like $\sqrt{2}$ cannot be represented as a fraction. Use a calculator to calculate $\sqrt{2}$. You will find its decimal representation never repeats and never ends. Hence it is irrational.'

'How many irrational numbers are there?'

'Oh Googol – there are many. Square root of all prime numbers is irrational, for example $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ and so on. Square roots of many composite numbers can also be irrational. For example, $\sqrt{39}$. However $\sqrt{16}$ is not irrational. Apart from them, there are a number of famous constants like Euler's number (e), pi (π), the golden ratio (Φ) – they are all irrational.'

'I have noticed many repeating decimal representations. For example, when I divide 2 by 3, result is 0.666....repeating, Does it mean 0.666.. is also an irrational number?'

'Good observation and the answer is hiding within your question itself. You said when you divide 2 by 3 you get 0.666. This signifies 0.666... could be represented as the ratio of two integers and hence it cannot be an irrational number.'

'So we have real and imaginary numbers. Within real numbers, there are rational and irrational numbers. Rational numbers can be integers of their ratios, called fractions. I guess there should be something more on irrational number'

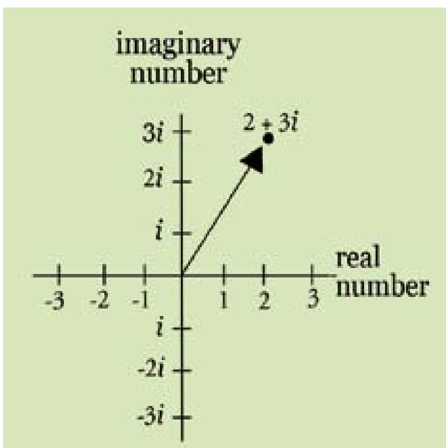
'Oh great! I think that your logic of symmetry is going towards the right direction. Within irrational number there are transcendental numbers'

'What is a transcendental number?'

'A transcendental number is a type of number that cannot be a solution of any polynomial equation.'

'Please elaborate, uncle.'

'Consider a polynomial equation: $x^2 - 2 = 0$. One of the solutions of this equation is $\sqrt{2}$. Therefore $\sqrt{2}$ is not transcendental number, although it is irrational. However, a transcendental number cannot be a solution of a polynomial equation of any order. '



'Please give some examples of transcendental numbers'

'Most of the famous constants like *e*, π, the golden ratio (Φ) are all transcendental'

'This means that all transcendental numbers are also irrational numbers'.

'You are absolutely right. All transcendental numbers are irrational, but not vice versa.'

'Yes I understand now. So, *e*, π, Φ are transcendental and also irrational. But the square root of 2 ($\sqrt{2}$) is irrational but not transcendental.'

'You're absolutely right!'

'Is there a name for those numbers that are irrational but not transcendental?' I asked.

'Yes, numbers that are irrational but not transcendental are called algebraic numbers. Therefore $\sqrt{2}$, $\sqrt{3}$ are all algebraic numbers.'

'How to prove an irrational number is transcendental or not?'

'It is not easy to prove that a specific number is transcendental. For this one must prove that the number does not fulfil a certain requirement. Among the numbers whose status has not yet been settled are π^e, π^π, e^e, and e^π.'

'Uncle, you have explained, within rational numbers, there are more fractions than integers. What about transcendental and algebraic?

'In 1847, German mathematician Georg Cantor made the startling discovery that there are more irrational numbers than rational ones, and more transcendental numbers than algebraic ones. In other words, most real numbers are irrational, and among irrational numbers, most are transcendental!'

'This is an amazing fact! There are infinite numbers and it is possible that there are many other facts that we do not know yet!'

'That's true, Googol!'

'There are real numbers and imaginary numbers. You have told me all about real numbers. Please tell me something about imaginary numbers.'

'Square root of any positive real number will give another positive real number. For example $\sqrt{2}$ = 1.414.. , $\sqrt{25}$ = 5. But if you want to do a square root for negative numbers, result will be an imaginary number. That means $\sqrt{-25}$ is an imaginary number.'

'How is an imaginary number represented?'

'An imaginary number can be written as a real number multiplied by the imaginary unit *i*, which is defined by its property $i^2 = -1$, or $i = \sqrt{-1}$. That means $\sqrt{-25} = i5$'

'I have heard about complex numbers. Are imaginary numbers complex numbers?'

'An imaginary number *ib* can be added to a real number *a* to form a complex number of the form $a + ib$, where *a* and *ib* are called, respectively, the real part and the imaginary part of the complex number. For example $2 + i5$ is a complex number.'

'Uncle, please tell me more about the complex number.'

This time my uncle interrupted me.

'My dear Googol, we will miss our morning yoga session if we talk more on numbers now. So let's concentrate on the yoga now and we will ponder over numbers sometime later.

A Primer on Prime Numbers

Recently my uncle got engrossed with the cases of Sherlock Holmes, the famous fictional detective created by the author and physician Sir Arthur Conan Doyle. Holmes was a London-based consulting private detective who is famous for his incisive and intelligent logical reasoning and forensic science skills to solve criminal cases. It was therefore not a surprise that these days my uncle kept asking me questions using his characteristic cryptic clues about facts related to Sherlock Holmes stories.

One evening, he told me that Holmes was featured in four novels and 56 short stories. The first novel, *A Study in Scarlet*, appeared in Beeton's Christmas Annual in 1887. All but four stories on Holmes are narrated by Holmes's friend and biographer, Dr Watson; two are narrated by Holmes himself (*The Blanched Soldier* and *The Lion's Mane*) and two others are written in the third person (*The Mazarin Stone* and *His Last Bow*). Conan Doyle wrote the first set of stories over the course of a decade. To devote more time to his historical novels, he wrote *The Final Problem* in 1893 when Holmes presumably died after the fall over Reichenbach Falls while fighting with his greatest opponent Professor Moriarty. After resisting public pressure for eight years, he brought back Holmes in *The Adventure of the Empty House*.

'Google, could you tell me which Sherlock Holmes' stories or novels had the following numbers in the title: 2, 3, 4, 5 and 6?'

'I can say, at least two of them. The number 4 appears in the second novel *The Sign of the Four*. The number 5 is in the story titled *The Five Orange Pips*.'

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Primes (marked red) in first hundred

'That's a good attempt, Googol. Let me give you the other answers. For the number 2, it's *The Adventure of the Second Stain*. There are at least three stories with number 3 in the titles. These are: *The Adventure of the Three Students*, *The Adventure of the Three Garridebs*, and *The Adventure of the Three Gables*. Finally, the number 6 appears in the title *The Adventure of the Six Napoleons*.'

'Yes, I read some of those stories,' I replied.

'Which address is also referred to as the world's most famous address?' uncle asked.

'It's 221B, Baker Street, London. It's the apartment where Sherlock Holmes lived.' I gave a quick reply.

'Fantastic, Googol. Do you know that this is indeed a real address in London? The complete address is: 221B Baker Street, London, NW1 6XE, England. According to the stories written by Sir Arthur Conan Doyle, Sherlock Holmes and Doctor John H. Watson lived at 221B Baker Street between the years 1881 to 1904. The house is now protected by the government due to its special architectural and historical interest, while the first floor study overlooking Baker Street is still faithfully maintained for the posterity as it was kept in Victorian Times.'

'That's amazing, Holmes is immortalised at 221B Baker Street. Uncle, is there anything special with the number 221?'

'Hmm, it's an interesting question. The number 221 is indeed a very fascinating number. I'll tell you more about this later. For the time being, I can say that the number 221 is a composite number which is the product of two prime numbers: 13 and 17.'

'Uncle, you just mentioned two unfamiliar mathematical terms: prime number and composite number. Could you please elaborate more on these terms?'

'A *prime number*, or a prime, is an integer greater than 1 that can be divided only by itself and 1. A natural number greater than 1 that is not a prime number is called a *composite number*. For example, 5 is a prime, as it is divisible by only 1 and 5, whereas 6 is composite, because it has the divisors 2 and 3 in addition to 1 and 6. This division between prime and composite numbers turns out to be one of the cornerstones of mathematics, and is a characteristic which is used in mathematical proofs over and over.'

'Are there limited number of prime numbers in number world?' I wanted to know.

'No. In fact, there are an infinite number of primes. Another way of stating this is that the sequence 2, 3, 5, 7, 11, 13, ... of prime numbers never ends. Most of the unsolved mysteries in mathematics are also related to prime numbers.'

'Is number 1 a prime?'

'*1 is not a prime number*. 2 is the first prime number and the only *even prime number*; *all other prime numbers are odd*,' uncle replied.

'Is zero a prime?' I wanted to know.

'It's a very good question, Googol. It is interesting to know that *zero is neither a prime nor a composite number*. It cannot be a prime because it has an infinite number of factors. It is not a composite number because it cannot be expressed by multiplying prime numbers. 0 must always be one of the factors.'

'Are all composite numbers formed by multiplying primes?'

'Yes. Let me explain. If we factorise a composite number into two smaller numbers, then it needs to be checked whether these two numbers are themselves primes or composites. For example, 6 factorises into 2 x 3. Both the numbers 2 and 3 are prime numbers. The number 18 factorises into 2 x 9. Here the number 2 is a prime but the number 9 is not. However, the number 9 factorises into 3 x 3 and the number 3 is a prime. Hence the number 18 can be written as 18 = 2 x 3 x 3. Any composite number, no matter how large, can be factorised into two smaller numbers. We then ask whether each of the smaller factors is a prime or composite. If either one is composite, we factorise it again. The process continues till all the factors are primes. This in itself is interesting and leads to a fascinating conclusion. When a composite number is factorised into primes, those primes are *unique* to that number. For example, we can factorise the number 30 into 2 x 3 x 5. No other set of primes, when multiplied together, will yield 30.'

'This is very interesting, uncle!'

'This interesting fact leads to one of the building blocks of mathematics, viz., *every whole number greater than 1 can be expressed as a product of prime numbers in one and only one way*, which has come to be known as the fundamental theory of arithmetic.'

'I understand now why the number zero cannot be a composite number. The number zero can be expressed as 0 = 0 x 2 x 3 or 0 = 0 x 7 x 17 or infinitely many different ways. A composite number can be expressed as a product of prime numbers in one and only one way. Hence the number zero cannot be a composite number. At the same time zero has infinite numbers of factors. Hence it cannot be a prime. However, I'm still unclear why 1 is not a prime. Could you explain this to me?'

'If 1 is considered a prime, then the fundamental theory of arithmetic breaks down! Because 30 = 2 x 3 x 5 and also 30 = 1 x 2 x 3 x 5. Hence factorisation of 30 will not be unique. Therefore, 1 is not a prime number,' uncle replied.

'Please tell me more about prime numbers. They seem to have many fascinating properties.'

Column 1	Column 2	Column 3	Column 4	Column 5	Column 6
1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36
37	38	39	40	41	42

'Prime numbers are randomly distributed among the natural numbers, without any apparent pattern. However, the global distribution of primes reveals a remarkably smooth regularity. If you arrange all positive integers in a table having six columns, then you will notice that all the primes, except 2 and 3, are either in column 1 or column 5. You can try to expand the table and you will note that all prime numbers, except 2 and 3, will appear either in column 1 or in column 5. However, the distribution of primes within column 1 and 5 is random,' uncle explained.

'It's very interesting, uncle. Please tell me when mathematicians knew about prime numbers.'

'It is believed that around 1650 BC, ancient Egyptians knew about the prime numbers. However, the earliest surviving records of the explicit study of prime numbers come from the Ancient Greeks. Euclid's *Elements* (circa 300 BC) contain important theorems about primes, including the fundamental theorem of arithmetic,' uncle replied.

Pierre de Fermat (1601- 1665)

'That means mathematicians knew about the prime numbers much before conceptualising the number zero!'

'Yes. After the Greeks, major documentary evidence of the study of prime numbers appeared in the 17th century. In 1640, French lawyer and an amateur mathematician Pierre de Fermat, while researching perfect numbers came up with a formula to generate prime numbers. Fermat conjectured that all numbers of the form ($2^{2^n}+1$), where n is any natural number, are prime. Fermat verified this up to n = 4 (or 2^{16} + 1). Prime numbers generated from this equation are known as *Fermat number*. Fermat could not verify it beyond n = 4. Later it was found that the very next Fermat number 2^{32} + 1 is a composite, 641 being one of its prime factors. In fact, no further Fermat number is known to be a prime.'

'Uncle, please elaborate on the *distribution of prime numbers*.'

'Prime numbers are not uniformly distributed. For example, between 1 and 100 there are 25 primes, while there are 21 primes between 101 to 200.'

'Does the frequency of prime numbers reduce as we move toward eds larger number group?"

'Yes, that too with some regular pattern. German mathematician and physical scientist Friedrich Gauss did significant work on prime numbers. Gauss spent hours trying to figure out some pattern or regularity in the distribution of prime numbers. Initially he confirmed the findings of the ancient Greeks that there appeared to be no pattern. However, later he discovered that if numbers are grouped according to powers of 10 (that is: 1-10, 1-100, 1-1000, etc) and then if one picks a number at random from within each range, the probability of it being a prime has some regular pattern.'

Carl Friedrich Gauss (1777-1855)

'It is fascinating to see the mathematical pattern in apparent randomness.'

'True. In this way regularity appeared out of the mist of

Number (up to...)	**No. of primes**	**Probability**
10	4	1:2.5
100	25	1:4
1,000	168	1:7
10,000	1,229	1:9
100,000	9,592	1:11

disorder. Each time a larger number group was considered, the probability of getting a prime number went down. That is, as the numbers got bigger, the prime numbers thinned out according to a predictable pattern. This eventually led to the *prime number theorem* (PNT) that describes the asymptotic distribution of the prime numbers. The prime number theorem gives a general description of how the primes are distributed amongst the positive integers. The prime number theorem states that the number of primes less than n is approximately n divided by the logarithm of n.'

'Is there any way to know what the n^{th} prime is, or do we have a simple formula to obtain a prime number in a sequence?' I wanted to know.

'Despite efforts by all leading mathematicians, there is no formula for computing the n^{th} prime. Many formulae do exist that produce nothing but prime numbers. However, these formulae do not produce each successive prime nor do they predict the next prime in sequence.'

'What is the largest known prime number?' I wanted to know.

'Till now, the *largest known prime* is $2^{43112609} - 1$. It is a *Mersenne prime*. French philosopher and mathematician Marin Mersenne showed that all numbers in the form ($2^p - 1$) are prime numbers, where 'p' is a prime. The largest known prime has almost always been a Mersenne prime.'

'Uncle, please tell me some applications involving prime numbers.'

'Primes, once the exclusive domain of pure mathematics, have recently found an unexpected ally in matters of computer security. Based on the difficulty of factorising a product of two very large primes, *public-key cryptography* was invented. *Public key cryptography* algorithms utilize prime numbers extensively. Prime factorisation is the key to all e-commerce applications, where financial transactions are done over Internet,' uncle replied.

'Please elaborate,' I said.

'When we exchange secret data, like bank account information, password, etc., there is a chance that a third person may intercept the data and may try to take undue advantage out of it. To protect secret data, *Public Key Cryptographic System* (PKCS) was developed. The system is based on prime numbers' uncle explained.

'Please tell me how prime numbers are able to protect secret data,' I wanted to know.

'Take two very large prime numbers, say P1 and P2. Multiply P1 and P2, say you get N, where N = P1 x P2. If I give you N and ask you to find P1 and P2, it would be difficult for you to find P1 and P2. Here P1 and P2 are unique to N, called prime factors. For example, consider P1 = 53 and P2 = 59, then, N = 53 x 59 = 3127. It is easy. However, if I give you 3127, and then ask you to find its prime factors, it would take some time before you get the answer. Using a computer program will be helpful. However, if P1 and P2 are very big prime numbers, say 150 digits each, then even a computer will take substantially long time to get the prime factors. This is the basis of PKCS.' uncle replied.

'I always wanted to know how e-commerce transactions take place over Internet. It seems mathematics is the answer!'

'Yes Googol. Before PKCS was invented, secret communication used to take place using secret codes. For example, if person *A* wants to send confidential data to another person *B*, both *A* and *B* will share the same secret code. *A* will encrypt the data using the secret code and *B* will decrypt using the same secret code. In its simplest form, say, *A* wants to send bank account number "1789" to *B*. *A* encrypts "1789" by multiplying it by 7, i.e., *A* sends "12523" to *B*, who already knows that 7 is the secret key. On receiving "12523", *B* will divide it by 7 and gets back the original number. However, the drawback of such system is that if *B* receives such communication from multiple persons, multiple secret codes will be required. It is analogous to buying a separate lock and key for each transaction. If *B* has done 10 transactions, *B* will need 10 keys. *B* has to protect and manage all 10 keys. Think about another situation, where *B* has only one key and multiple similar types of locks. *B* distributes these locks to all he/she wants to do transactions and keep the key with him/her. Everybody will encrypt the secret data using the lock *B* has provided and send it back to *B*. Now, as *B* has the key, only he/she will be able to decrypt all information. Note, *B* has to protect and manage only one key. Isn't it simpler?'

'Uncle, please elaborate how these lock and keys are implemented in mathematics.'

'PKCS is based on a pair of keys, called *private key* and *public key*. *Public key* is analogous to lock and *private key* is the secret code, as I have just explained. Secret information is encrypted using *public key* and decrypted using *private key*. PKCS is implemented mathematically using an algorithm, called RSA algorithm, named after its inventors Ron Rivest, Adi Shamir and Leonard Adleman.'

'How are private and public keys generated and how are prime numbers involved?' I wanted to know.

'Suppose *B* takes two primes *P1* and *P2*. Multiply *P1* and *P2* to get *N*, $N = P1 \times P2$. Now, *public key* is (*N, e*), where, *e* is a small public exponent. This *public key* of *B* is known to everyone. *Private key* is (*N, d*) and is known to *B* only. *d* is another number calculated from *P1* and *P2* using some mathematical function. Consider *A* has to send secret information '*m*' to *B*. *A* will encrypt '*m*' using the *public key* (*N, e*). Let '*c*' be the encrypted data, derived from '*m*', *N* and *e*. *B* will receive '*c*'. Using the *private key* (*N, d*), *B* will be able to decrypt the secret information '*m*'.'

'Uncle, I have a question. When *A* is sending '*c*' to *B*, there is a possibility of an unauthorised person intercepting it. Can that unauthorised person decrypt the secret code if he/she has the *public key*?'

'Googol, the public key is available to everybody, including the unauthorised third person. However, the encrypted data '*c*' can only be decrypted by the private key, which is known only to *B*,' uncle replied.

'Is it possible to guess the private key from the public key? After all, both the *private key* and the *public key* are generated from two prime numbers *P1* and *P2*,' I wanted to know.

'It is almost impossible to guess. If *N1* and *N2* are large prime numbers, even a supercomputer will take thousands of years to guess *private keys* from *public key* and the *encrypted data*. Therefore PKCS is very safe.'

'But I have heard about fraudulent practices over net

banking and similar e-commerce transactions! How is it possible?' I wanted to know.

'These do not happen due to failure of PKCS. RSA algorithm can never fail to provide adequate security. All the fraudulent practices that are reported are due to careless mistakes of the people involved in it – like sharing user ID, password, etc.'

'Thank you for explaining. Coming back to our Sherlock Holmes query, what's the special thing about the number 221?' I asked.

'In mathematics, a *semiprime* (also called *biprime* or *2-almost prime*, or *pq number*) is a natural number that is the product of two (not necessarily distinct) prime numbers. As I mentioned before, 221 is the product of two prime numbers, 13 and 17. So, 221 is a semiprime number. Examples of a few other semiprime numbers are: 4, 6, 9, 10, 14, 15, 21, 22, 25 and 26.'

'That's very fascinating indeed. What are the characteristics of the semiprime numbers?'

'The square of any prime number is a semiprime, so the largest known semiprime will always be the square of the largest known prime, unless the factors of the semiprimes are not known. As you could guess, based on the Mersenne prime, the largest known semiprime is $(2^{43112609} - 1)^2$, which has over 25 million digits. Like prime numbers, the semiprime numbers are also very important for cryptography and number theory.'

'It's not a wonder that 221B Baker Street had the most worthy inhabitant there – Sherlock Holmes.'

'The number 221 also has other attractive features. It's also the sum of five consecutive prime numbers (37 + 41 + 43 + 47 + 53 = 221) and the sum of nine consecutive prime numbers (11 + 13 + 17 + 19 + 23 + 29 + 31 + 37 + 41 = 221).'

'That's amazing, uncle! After knowing about the prime numbers and Holmes, I was thinking how the modern Holmes will look today.'

'Go on Googol.'

'Apart from his immense knowledge on botany, geology, anatomy, chemistry and forensic science, Holmes is a cryptography specialist as well. He reads the morning newspaper, browses the Internet, tweets, texts and blogs regularly. Taking the cues from the modern day technologies, he solves the cases in his mind in no time. I am though not sure how much he likes to be in the glare of social or electronic media!'

'It's elementary, my dear Googol,' uncle said in a tone similar to that of Sherlock Holmes.

A Tale of Two Digits

It was a Friday evening. It was also a relaxed evening for me. There was no rush for the homework to finish for next day's school. I was trying to decipher some puzzles from a book. My Uncle was going through the day's newspaper for the last time.

"Googol, if you have nothing to do now, let's play Botticelli," Uncle told me.

"What type of game a 'Botticelli' is?" I gave a puzzled look at Uncle.

"Botticelli is a guessing game in which players guess the identity of a person based on his or her biographical details using 'yes' or 'no' replies."

"That sounds very interesting though the name of the game is a bit strange!"

"The name was given after Sandro Botticelli who was an Italian painter of the Early Renaissance. The game takes its name to suggest that the famous person has to be at least as famous as Sandro Botticelli. "

"Hmm, I did not know about Botticelli until now; it seems that I might not be very good at this game," I confessed.

"The game of Botticelli has different variants. But the common theme is that one person or team thinks of a famous person, reveals his or her initial letter, and then answers 'yes' or 'no' to different statements allowing other players to guess the identity."

"I got it now, let's play the game."

"Well, we will play it a bit slightly differently. I've written the name of a person on this paper and I'll not give you any hint through the initial. You will tell me the statements, and I will give you a 'yes' or 'no' reply." Uncle explained showing me a folded piece of paper in his hand.

"I understand now," I replied.

"Well, then let's start," my Uncle was quite eager to know how I play the game of Botticelli.

"The gender of the person is male," I said.

"Yes," Uncle replied.

"He is an Indian."

"Yes."

"He is still living with us."

"No."

"During his lifetime, his activities spanned around the post-Independent India."

"No"

"He was involved with struggle for India's independence"

"Yes."

"He was behind the non-violent civil disobedience movement," I tried to focus on a target.

"Yes."

"He led in the Salt Satyagraha, Non-cooperation movement and Quit India movement'

"Yes, I think that you got it now."

"Mahatma Gandhi," I said emphatically.

"Well done, Googol," said Uncle as he showed me the unfolded paper with 'Gandhiji' written on it. I smiled.

"There must be similar games like this one where players have to guess things other than the famous persons,' I was curious to know.

"You are right. There are several flavours of games similar to Botticelli. For example, 'Vermicelli', in which the thing to be guessed is a food; 'Vespucci', in which the thing to be guessed is a place; and 'Webster', a challenging variant in which the thing to be guessed can be any word."

"Oh, that's a good range of games indeed. We can play all those games for a full day!"

"Yes, we may give them a try some day."

"The framing of the statements is crucial to this game and one needs to have a good biographical knowledge too. Only a 'yes' or 'no' answer would lead to the solution, that's quite interesting!"

"Well, here's another thought. Did you realise that at times literature and mathematics converge? Can you tell me the connection of these words with the number system: yes/no, true/false or presence/absence?"

"These are opposites in meaning and representing two states of an event."

"That's a good interpretation, Googol. Mathematically, we can define them as two states of a *binary variable*. In the mathematical world, we can also translate those words into two numeric values, 1 and 0."

"So, the numbers 1 and 0 represent a special type of number system."

"Yes, this is also called *binary number system*," Uncle replied.

"I have heard about the binary numbers, but I do not have a clear idea about what they signify. Please explain this to me."

"Before that, here is a riddle for you. What is the link between Mahatma Gandhi and binary numbers?"

It must be one of those characteristic riddles from Uncle. I was perplexed and did not have a clue.

"Well, here is another clue. The link is hidden in the *Gandhi Jayanti* and *International Day of Non-violence*."

"I know that. Gandhiji's birthday, i.e., the second of October is commemorated as *Gandhi Jayanti* in India and world-wide as the *International Day of Non-violence*."

"And if you write down the date using a date format, what will you get?"

"It will be the second of October, or 2/10."

"Exactly, I hope that you can see the link now. The word 'binary' means the number system is represented by two numeric values and these two numbers are 1 and 0. Moreover, binary representation of 2 is 10. So the date '2/10' in essence captures the concept of binary number."

"Oh yes, I can see the link now. Uncle, please tell me why the binary number system is so important for us."

"The binary numbers form the basis for the operation of computers and all digital circuits. As I mentioned earlier, any number can be represented in a binary number system using different combinations of two numeric symbols, 0 and 1."

"That sounds very interesting – any number can be formed by using only two numeric symbols."

"Tell me, how many symbols do we use in the decimal number system, i.e., the number system that we generally use for writing numbers?" Uncle wanted to know.

"Ten symbols – zero to nine," I answered.

"That's right. The decimal number system uses ten *symbols* 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 to represent any number. These symbols are called *digits*. This system is used worldwide as the most convenient system to represent numbers. You may remember that we talked about this earlier. Rules for operation of 'zero' were given by Indian mathematician *Brahmagupta* during AD 600. The invention of 'zero' made it possible to write numbers with positional values."

"What is a positional value?" I wanted to know.

Decimal number: 2 3 5

$$2 \times 10^2 + 3 \times 10^1 + 5 \times 10^0$$
$$= 200 + 30 + 5$$
$$= 235$$

Binary number: $(1\ 0\ 1)_B$

$$1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$$
$$= 4 + 0 + 1$$
$$= (5)_D$$

"When we write a decimal number, say 235, then 2, 3 and 5 are not merely symbols. Each position of the number has a *positional value*, for example, the first place has a positional value of 1; the second place has a value of 10; the third place of 100 and so on. The digits are multiplied by the positional values and then added up to represent a number. Hence the number 235 signifies: $2 \times 100 + 3 \times 10 + 5 \times 1 = 235$, " Uncle explained.

"Yes, I got it now, any number in the decimal system can be written according to the positional value of each symbol."

"Now, if you look into the above expression closely, you will notice that all the positional values can be represented as 10^n form, where n is any *positive integer*. Like, $10^0 = 1$, $10^1 = 10$, $10^2 = 100$ and so on. For example, the above expression of 235 can also be written as: $2 \times 10^2 + 3 \times 10^1 + 5 \times 10^0 = 235$."

"Is it due to the fact that there are ten symbols in the decimal number system?"

"That's a good observation, Googol. Yes, you are right. For the decimal number system 10 is called the *base* as there are ten distinct symbols. However, note that n can also be a *negative integer*. When we write a number with decimal point, say 273.45, the positional values at the right hand side of the decimal point will be 10^{-1}, 10^{-2} and so on. Therefore, $273.45 = 2 \times 10^2 + 7 \times 10^1 + 3 \times 10^0 + 4 \times 10^{-1} + 5 \times 10^{-2}$."

"Please explain to me how the binary numbers are formed," I wanted to know.

"In the binary number system, there are only *two symbols* 0 and 1. Or you can say that we use only *two digits* (0 and 1) of the decimal number system. These are called binary digits or bits. Hence, the *base of the binary number system* is 2. The *positional values* are in 2^n form, where n can be *positive or negative integers*. For example, the decimal equivalent of the binary number 101 is: $1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 5$. The decimal equivalent of the binary number 101.11 is: $1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 + 1 \times 2^{-1} + 1 \times 2^{-2} = 5.75$."

Decimal	Binary	Octal	Hexadecimal
0	0	0	0
1	1	1	1
2	10	2	2
3	11	3	3
4	100	4	4
5	101	5	5
6	110	6	6
7	111	7	7
8	1000	10	8
9	1001	11	9
10	1010	12	A
11	1011	13	B
12	1100	14	C
13	1101	15	D
14	1110	16	E
15	1111	17	F
16	10000	20	10

"It does not look very difficult! Can we convert a number from the decimal system to the binary system? "

"Yes. For that you have to keep dividing the decimal number by 2 till you get a *quotient* 0. Note, each time you divide a decimal number by 2, you will get the remainder 0 or 1. Now write all the remainder in reverse order and this will give you the binary equivalent of a decimal number. For example, consider a decimal number 12. Now, 12 / 2 = 6 (quotient) and the remainder is 0. Next, divide the quotient 6 by 2. This will be 6 / 2 = 3 and the remainder is 0. Next, 3 / 2 = 1 and the remainder is 1. Next, 1/2 = 0 and the remainder is 1. Now write all the remainders in the reverse order, which is 1100. Therefore the binary equivalent of 12 is 1100."

	Quotient	Remainder
12/2	6	0 ↑
6/2	3	0
3/2	1	1
1/2	0	1
Binary equivalent of 12 is 1100		

"I will try to convert a few numbers from the decimal to the binary and also the binary to the decimal system later using the above strategies. Now, please tell me more on how the binary number system was developed."

"The Indian scholar Pingala (2nd centuries BC) developed the mathematical concepts for describing poetry, and thus presented the first known description of a binary numeral. He used binary numbers in the form of short and long syllables (long syllables is equal to two short syllables), making it similar to the Morse code," Uncle said.

"I have heard that the Morse code is used to transmit information using short and long pulses of sound or light. Does it follow the binary number system?"

"If the information can be coded using two distinct states, it is indeed a binary system. Hence, the Morse code is a binary code."

"Like many other mathematical discoveries, the binary number system was also developed in India – I feel very proud of being Indian."

"That's right. In his Sanskrit classic *Chhanda Sūtra,* Pingala described a method to assign a unique value to each line, something very similar to the binary number system. This is perhaps the oldest description of something similar to the binary number."

"Please tell me what happened after that."

"During the 11th century, the Chinese scholar and philosopher Shao Yong developed a system similar to modern binary number system. Shao Yong's work influenced the German mathematician and philosopher Leibniz in the 17th century in formulating the binary arithmetic."

"Was Shao Yong's work influenced by Pingala's *Chhanda Sūtra*?" I wanted to know.

"It appears that during the 10th century, another Chinese scholar Yang Hsiung developed a number system with the base 3. Shao Yong was influenced by this number system and developed the number system with base 2."

"How did Leibniz develop the binary arithmetic?" I wanted to know.

"Leibniz was aware of the work of Shao Yong on the binary numerals. He noted that the hexagram used by Shao Yong corresponded to the binary numbers from 0 to 111111 and this mapping is one of the major milestones in formulating binary number system. Leibniz also used 0 and 1 as binary numerals."

"How was binary number system introduced in the computer?"

"Computer is based on the digital system. The digital system is based on the binary logic, having two distinct states, TRUE and FALSE. Therefore we can conclude that the binary number system actually triggered the development of computer. "

"It is very interesting – please tell me more on this."

"Any computer system works on the *binary logic* – logic 0 or FALSE and logic 1 or TRUE. Logic 0 and 1 are *two distinct states*, represented by two voltage levels, 0 volt and 5 volts respectively. You can imagine without the binary logic, it would have been very difficult to generate and decipher multiple voltage levels to store and interpret information. Remember, the computer is not a human – it recognises data differently from us. All data in a computer are only recognised by appropriate electrical signals."

"I thought the binary numbers are required for mathematical calculations only. It seems that storing and processing of data also need the binary number system. I am very keen to know more about it. Please explain how the data are stored in a computer. "

"Any data consist of alphabets and numerals, called *alphanumeric* characters. Each of these characters is assigned a distinct numeral value. For example English alphabet 'a' has assigned number 97. Its binary representation is 1100001. Similarly other characters have other distinct values. These binary numbers are stored in the *computer memory*. "

"How are arithmetic operations performed in a computer?" I wanted to know.

"Calculations in a computer are based on the *Boolean algebra* – operations like addition, subtraction, multiplication and division are carried out on the binary numbers," Uncle replied.

"Please tell me something about the Boolean algebra.'

"In 1854, the British mathematician George Boole published a landmark paper that described arithmetic operations on the binary numbers. The Boolean algebra is quite similar to the algebra you do on the decimal numbers. "

George Boole

1815–1864

"Uncle, how does a computer take a decision?"

"As I mentioned earlier, a computer cannot take decisions as a human does. Computers can only check whether a condition is TRUE or FALSE – called logical operation. Based on a series of logical operations, a computer analyses the data and provides an output. ”

“What is the logical operation?” I wanted to know.

“In a *logical operation*, a computer verifies some conditions. If the condition is TRUE, it performs some operations, else it performs some other operation. For example, if *a* is greater than *b* then add *a* and *b*, else (i.e. if *a* is less than or equal to *b*), subtract *a* from *b*. The entire computer operation is based on the execution of this kind of simple logic.”

“That’s very interesting. Please tell me how logical operations are performed.”

“Logical operations are also performed using the Boolean algebra. Like algebraic operators (+, –, ×, /) there are *logical operators* ‘OR’, ‘AND’, ‘NOT’, etc.”

“It’s very interesting. It is clear to me that the development of computer would not have been possible without the binary number system.”

“You are right, Googol. In 1937, the American mathematician and electronic engineer Claude Shannon, while working on his thesis at MIT, implemented the Boolean algebra and binary logic using electronic relays and switches. Shannon’s thesis essentially founded the practical digital circuit design and eventually the computer was conceptualised. ”

“Uncle, I have heard about the octal and hexadecimal number system. Are they related to the binary number system? ” I wanted to know.

“The *octal number* system uses base 8 and the *hexadecimal number* system uses base 16. Note that both 8 and 16 can be represented as 2^n, where $n = 3$ in the octal and $n = 4$ in the hexadecimal system. It is therefore clear that both are related to the binary number system.”

"If the binary number system is everything for a computer, then what is the need for other number systems?" I wanted to know.

"That's a good question. Let me clarify. We, humans, use decimal system because positional value based on 10^n is easier to comprehend and calculate, and we have 10 distinct symbols. Higher order base will need more distinct symbols; moreover, it would make calculations difficult. A computer system uses the binary number system as it needs only two distinct states (electrical signals) to represent any character. The octal and hexadecimal number systems are just the extensions of binary number system for easier representation of binary numbers, so that we can represent larger binary numbers with smaller octal or hexadecimal numbers."

"That means a computer does not use the octal or hexadecimal number system!"

"Precisely that is the case. These two systems are used only for convenient representation of the binary numbers for our understanding. The binary numbers are easily converted to and from the octal numbers. The *octal number* system has eight distinct symbols 0 to 7. If you look closely, you will note that three binary digits are equivalent to one octal digit. For example, the binary number '110101' is represented as '65' in the octal number system, where 6 = 110 and 5 = 101."

"Please tell me about the hexadecimal number system – I was wondering what 16 different symbols will be in this number system?"

"In the *hexadecimal number* system, 16 distinct symbols are used. These are 0 to 9 and then *A, B, C, D, E* and *F*, with *A* equal to decimal 10, *B* equal to decimal 11 and so on. Here, four binary digits are equal to one hexadecimal digit. For example, the binary number '10011100' can be written as '9C' in the hexadecimal number system, where 1001 = 9 and 1100 = 12 = C."

"What is the advantage of having the octal and hexadecimal number systems?"

"As I have mentioned earlier, it makes the representation of binary numbers a lot easier. During the early development phase of the computer, the hexadecimal codes were entered as instructions. You can imagine, entering a binary code as instruction will make the life very cumbersome."

"Do we still use hexadecimal numbers as instruction to a computer?"

"In general, instructions are written in English like language, called the *high level language*. These instructions are converted to a hexadecimal code and eventually a binary code is generated. The binary code is finally deciphered by the computer. However, there are some applications where the hexadecimal codes are entered directly."

"Uncle, I have realised now that the binary number system is not merely a method to represent numbers; rather it is the basis of the entire computer applications."

"You are right. All the advancement of computers and associated applications would not have been possible without the binary numbers."

"The world of binary number is indeed amazing! Uncle, thank you very much for introducing me to this amazing world!"

"Yes Googol. Next time when you'll switch on a computer, I hope that you'll appreciate how the magic of mathematics is playing an important role behind the scene!"

"It's like the computer playing an infinite number of Botticelli games with us!"

"That's a good analogy, Googol."

A Chronicle of Complex Numbers

These days, I found that my uncle was very much occupied with the world of art. Recently there were several art exhibitions around the city and I accompanied him to a few of those. While I do not know much about art, I could see that my uncle was examining with meticulous care the details of all sculptures and paintings.

One evening, my uncle showed me a painting titled 'The Scream'. He said that the name is given to four versions of a composition, created as both oil paintings and pastel work, by the Expressionist artist Edvard Munch between 1893 and 1910. All works titled 'The Scream' depicts a figure with an agonised expression against the landscape of a turbulent red sky in a city in Norway. He told me the story about how the painting was born out of Munch's own experience one evening while he was walking along a path. The setting of blood red Sun in the backdrop of the landscape of the city amazed him and he sensed an infinite scream passing through nature. And 'The Scream' was born.

“'The Scream' could be an example of how to understand art. Examine the subject, identify the intricacies and complexities of objects around the subject, look for the style used by the artist, get the symbolic meaning of the composition being conveyed and so forth. Like any form of literature, art should be read and enjoyed in all its intricate details. That's why Vincent van Gogh, Pablo Picasso, Paul Cézanne, and others are so famous." uncle was trying to give me a flavour of understanding art.

“Sometime the abstract nature of art needs a lot of imagination," I said, as my little brain was trying to grasp some ideas on art.

“Googol, you would be surprised to know that a recent art exhibition in London was called 'Invisible: Art about the Unseen', which among its exhibits had pieces of papers with images drawn with invisible ink, blank paper titled '1,000 Hours Staring', a plinth titled 'Invisible Sculpture', and so forth. The very abstract nature of the exhibition is construed as meditation on seeing," my uncle went on.

“That's a lot of imagination indeed in the art world. At least, in the field of mathematics, many topics are not left to imagination," I remarked.

“Well, there is something called *imaginary numbers* in the mathematical world. But you are right – no stretch of imagination is associated with an *imaginary number*," my uncle quipped.

“Yes, I heard about this. The *imaginary number* is a part of the complex number. But uncle, I don't know much about this. Could you please tell me something more on this?" I was eagerly waiting for a breeze of mathematical ideas from my mathematician uncle.

“A *complex* number is a composite of real and imaginary numbers. Complex numbers are useful quantities that can be used in calculations. Imaginary part of a complex number is not left to the imagination of any individual and has well formulated rules. Calculations involving complex numbers produce meaningful results. However, recognition of this fact took a long time for mathematicians to accept," uncle said.

“It appears very complicated," I opined.

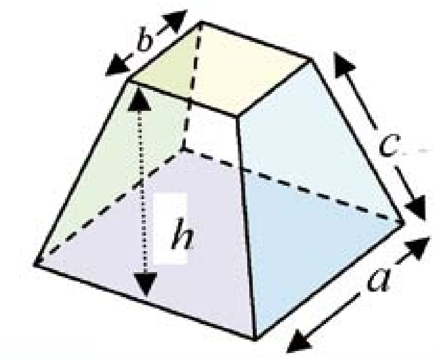
“The term 'complex number' does not mean that it's an intricate or complicated topic in mathematics. It means that there are the two types of numbers in the mathematical world, real and imaginary, and a complex number is formed by using both real and imaginary numbers together."

“But how do I visualise an imaginary number? I know about real numbers and the number line where rational and irrational numbers can be shown. Is there an imaginary number line also?"

“Indeed, there is an imaginary number line. The imaginary line perpendicularly bisects the real number line. Both number lines intersect at zero."

“Please tell me how mathematicians conceptualised the idea of some kind of number that is imaginary?"

“It appears that during first century AD, mathematician and engineer Heron of Alexandria tried to solve a problem related to measurement of volume of a truncated square pyramid, also called *frustrum* of a pyramid. Volume of such *frustrum* of a pyramid is given by $V = 1/3\ h\ (a^2+ab+b^2)$, where a and b are edge lengths of the bottom and top squares respectively, and h is height."



c = slant edge length

“But all numbers are real here, as we will be able to measure a, b, and h easily."

“That's true. But if the pyramid is a solid one, you have to measure h mathematically, as you won't be able to measure it directly," uncle replied.

“Can't h be measured from the *slant* edge length?' I wanted to know.

“You are right and that's what Heron calculated. He calculated $h = \sqrt{c^2 - 2\left(\frac{a-b}{2}\right)^2}$, where c is the *slant* edge."

"But I'm still unable to see any complex number here," I confessed.

"Okay Googol. If you put a = 28, b = 4 and c =15 in this formula for h, you will get $h = \sqrt{81 - 144}$ ".

"I got it now! You cannot get square root of a negative number!" I exclaimed.

"Well, it's not that you cannot get that, but what you will get is an imaginary number. $\sqrt{81 - 144} = \sqrt{-63}$ can be written as $\sqrt{-1} \times \sqrt{63} = i\sqrt{63}$ =," uncle replied.

"That means Heron of Alexandria discovered imaginary number."

"Not really. There was no concept of negative numbers, let alone square root of negative numbers during his time. Heron simply wrote $h = \sqrt{63}$ instead of $\sqrt{-63}$. Thus Heron missed being the earliest known mathematician to have derived the square root of a negative number."

"When did mathematicians find the correct solution?" I wanted to know.

"Astonishingly it took many centuries to recognise square root of a negative number."

"Why did it take so long time?"

"Ancient mathematicians rejected negative numbers, as they could not physically interpret a number that is less than nothing, as zero was considered absence of anything. During the third century AD, Mathematician Diophantus, Heron's fellow Alexandrian, also completely missed the opportunity of discovering imaginary numbers only because he did not accept the idea of a negative number. In his book *Arithmetica*, Diophantus stated the following problem:

Given a right angled triangle with area 7 and perimeter 12, find its sides.

To solve it, Diophantus derived the quadratic equation $336x^2 - 172x + 24 = 0$. If you solve this quadratic equation, there will be two solutions for x and both will involve complex numbers. However, Diophantus ignored the negative sign within a square root and considered only positive root in solving the equation."

"Uncle, please wait a minute. I can solve this equation! Solution of a quadratic equation of the form $ax^2+bx+c = 0$ is given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

So, here the term (b^2-$4ac$) will be: (172^2 – 4 × 336 × 24) = –2672. This will result in the square root of a negative number," I said after doing a quick calculation on my hand calculator.

"Yes, you got it right Googol. But my dear, you need to understand that we are talking about mathematics of the third century."

"I understand, the number system was not so well-developed at that time and the concept of negative number was not there at all. Please tell me what happened next."

"In AD 850, Indian mathematician Mahaviracharya gave some hints of a negative number," he stated:

"The square of a positive as well as of a negative is positive; and the square root of those are positive and negative in order"

"Mahavira was so close!"

"Indeed, Mahavira was very close in defining negative numbers. But he did not accept square root of negative numbers," uncle replied.

"When did mathematicians calculate square root of a negative number?"

"Actually the study of complex numbers began during the 16th century when algebraic solutions for the roots of cubic and quartic polynomials were discovered by Italian mathematicians."

"Uncle, please elaborate more on this. I thought the concept of $\sqrt{-1}$ came from quadratic equation $x^2 + 1 = 0$, where $x^2 = -1$, hence, $x = \sqrt{-1}$"

"This equation can define $\sqrt{-1}$, but the concept of complex number came from an entirely different situation!"

“Please tell me, it's really very exciting," I said.

“In 1494, Italian mathematician Luca Pacioli wrote a book '*Summa de Artthmetica, Geometria, Proportioni at Proportionalita*', summarising all the knowledge of arithmetic, algebra and trigonometry. He declared that the solution of a cubic equation was impossible with the knowledge that existed during his time. However, within ten years, another Italian mathematician Scipione del Ferro solved the *depressed cubic*."

“What is a *depressed cubic*?" I wanted to know.

“The general cubic is $x^3 + a_1x^2 + a_2x + a_3 = 0$, which contains all the powers of x. In a *depressed cubic,* the second degree term is missing, i.e., $x^3 + a_2x + a_3 = 0$. Solution of this form of an equation was very important. It is considered as the first step in defining the square-root of minus one, and thereby, imaginary numbers."

“Who introduced the symbol i for $\sqrt{-1}$?" I wanted to know.

“In 1777, the Swiss mathematician Leonhard Euler introduce the symbol i".

“There was a long gap between Del Ferro's solution and recognition of $\sqrt{-1}$!"

“Yes. Del Ferro and his fellow mathematicians were looking for only single real positive numbers for the solution of a cubic. They did not see that there were two complex roots. For example, the equation $x^3 + 6x = 20$ has one real solution $x = 2$. However, it also has two complex roots $(-1 + 3i)$ and $(-1 – 3i)$."

“That means Del Ferro did not discover complex numbers?"

“No single mathematician can be credited for the discovery of complex numbers, as many contributed directly and indirectly in understanding the nature of complex numbers. Another Italian mathematician Niccolo Fontana, known as Tartaglia (“the stammerer") also solved cubic equation independently. In 1545, mathematician Girolamo Cardano published a book,

Niccolo Fontana
(1499 - 1557)

Girolamo Cardano
(1501 – 1576)

Ars Magna (The great art), and gave credit to both Del Ferro and Niccolo Fontana for the solution of *depressed cubic*. Cardano also showed how to extend the solution of *depressed* cubic to all cubics."

"So the solution provided by Cardano could solve all cubic equations?"

"Yes, you're right. It was proved later that Cardano's method could solve any cubic equation. Some cubic equations yield only real roots while some yield real as well as imaginary roots."

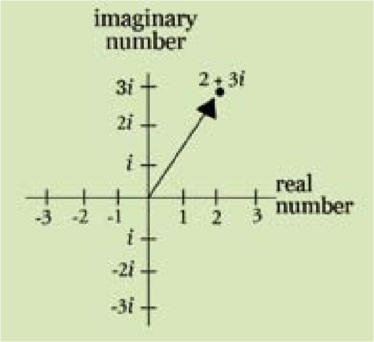
"Did Cardano mentioned anything about $\sqrt{-1}$?" I wanted to know.

"He mentioned about it unknowingly. One problem in his *Ars Magna* was:

Dividing ten into two parts whose product is forty. This problem leads to the quadratic equation $x^2 - 10x + 40 = 0$. Solution to this equation is $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$, which Cardano called *sophistic*. Their sum is obviously 10 and product was calculated as:
$(5 + \sqrt{-15})\ (5 - \sqrt{-15}) = (5)(5) - (5)(\sqrt{-15}) + (5)(\sqrt{-15}) + (\sqrt{-15})\ (\sqrt{-15}) = 25 + 15 = 40$."

"It's surprising! It's actually an algebra using complex numbers," I wondered.

"Yes, you got it right Googol. However, Cardano did not formulate any algebraic formula using complex numbers. It was an Italian engineer and architect Rafael Bombelli, who eventually discovered how to calculate the root of a complex number using Cardano's formula."



"Why is the non-real part of a complex number called imaginary?" I wanted to know.

"Before Cardano, there were several terms used for the imaginary part of the complex number, like impossible number, improbable number, nonsense number, etc. Cardano himself used the term sophistic. The great French thinker Rene Descartes dismissed it as an 'imaginary' number which was not a complimentary statement at that time, but the name stuck in the world of mathematics."

Rafael Bombelli (1526 – 1572)

Caspar Wessel (1745– 1818)

"Can complex numbers be represented using geometry?" I wanted to know.

"It's a good question. In fact Bombelli's formal meaning to $\sqrt{-1}$ still did not include the physical interpretation. Mathematicians tried to formulate the geometric meaning of $\sqrt{-1}$. A Norwegian surveyor Casper Wessel constructed graphical representation of complex number with the introduction of imaginary axis and complex plane. That is if a and b are both real then $a + b\sqrt{-1}$ is complex. $\sqrt{-1}$ is generally represented as i and $a + ib$ is a common form of representing a complex number, where a is real part and ib is imaginary part. Wessel was not a professional mathematician, but a very well known surveyor. In his paper (1797) 'On the Analytic Representation of Direction: An Attempt', to the Royal Danish Academy of Science, he introduced writing a complex number as $a + ib$."

"Is Gaussian plane also a complex plane?" I wanted to know.

"That's right Googol. The German mathematician Johann Carl Friedrich Gauss made significant contribution in the geometric interpretation of complex numbers. In his honour, the complex plane is also called Gaussian plane. After Gauss's interpretation of the complex number, $\sqrt{-1}$ was accepted as a legitimate symbol."

Carl Friedrich Gauss (1777 – 1855)

"Can a complex number have only imaginary part?" I wanted to know.

"Certainly yes. If real part is zero, a complex number will be left with only imaginary part," uncle replied.

"What about arithmetic operation involving complex numbers?"

"There are set rules of addition, subtraction, multiplication and division of complex numbers. These rules are not very different from rules involving real numbers."

"Please tell me some applications of complex numbers," I told.

"As we discussed before, the concept of complex number is required to solve polynomial equations. Polynomial equations are constructed for mathematical modelling of many physical phenomena. The complex number system is also embedded in other branches of mathematics like geometry, trigonometry and calculus and therefore its areas of applications are manifold. Complex numbers play an important role in understanding many phenomena in physics and astronomy. Apart from the pure science, complex number is an integral part of electrical engineering to understand the circuit behaviour, electromagnetism, and control circuit feedback mechanism. In telecommunication, the signal processing and synthesis cannot be done without a fair knowledge of complex numbers. The complex number system is also an intrinsic part of quantum mechanics. There are several other applications of the complex number and the list is indeed very long."

“Uncle, thanks very much for telling me such a fascinating story of the complex number and its extensive applications. It was an enjoyable journey through the history of complex number."

“That's good. Well Googol, it's almost the dinner time now. Let's get ready for the dinner."

Suddenly I realised that I was feeling hungry, and it was certainly not my imagination!

The Calculus Affair

It was a lazy Sunday morning in the study room. I just finished my homework. Uncle was still engrossed with his morning newspaper. I did not wish to disturb him. So I turned to the sports page of the newspaper. The London Olympic was the big news. Reports covered success stories of Olympians Vijay Kumar, Sushil Kumar, Yogeshwar Dutt, Gagan Narang, MC Mary Kom, Saina Nehwal and Paralympian Girisha Nagarajegowda – it was indeed very proud moments for every Indian.

“What's the link between the numbers: 10^0, 9.58, 9.63 and 10^2?" uncle asked without looking away from the newspaper. The question was certainly directed towards me.

I was slightly perplexed. 'They are all rational numbers,' I tried to reason.

“A letter or word clue now: I, WR, OR, and C," uncle could perceive my hesitation.

“Roman numerals I is for 1 or 10^0, C is for 100 or 10^2. Now, WR and OR...wait a second...World Record and Olympic Record...Usain Bolt." I almost screamed.

“Fantastic. Yes, the Jamaican sprinter Usain Bolt is the fastest man on the earth holding the 100 metre World Record (9.58s) and Olympic Record (9.63s)." My uncle put the rejoinder.

“And he is also the first man in the history to achieve 'double double' by winning both the 100 metre and 200 metre Olympic sprint titles in successive two Olympics - in Beijing and London," I added.

“That's right. Don't forget that behind these achievements, there are hundreds of hours of discipline, dedication, devotion and determination. Put yourself in a situation when hours of preparations must be executed in less than 10 seconds. That's a real inspiration for everyone!" Uncle said.

"And someone is repeating the same feat two times in a row – that's unbelievable!"

"You are quite right Googol."

"At least the mathematicians are in a better job – they can always repeat their performance with perfection, can't they?" I joked.

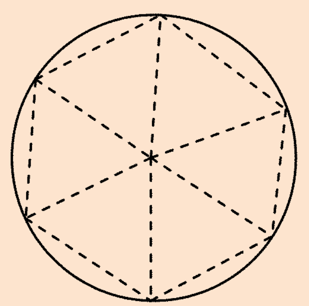
"That's fair to say, Googol. But it might not be always true. Well, here is riddle for you. When does a mathematician say: mathematics is painful?"

"Mathematicians feel mathematics is painful!" I was surprised.

"The answer is: Calculus," uncle quipped seeing my puzzled face.

"I know calculus is a branch of mathematics, but I did not get why a mathematician feels calculus is painful," I was still baffled.

"My dear Googol, you did not get the pun. Etymologically, the word 'calculus' originated from the Greek word 'kalyx' meaning the pebble or small stone. The original meaning of this word is still retained in the medical science where it means a stone, or concretion, formed in the gallbladder, kidneys, or other parts of the



The area of a circle is approximately equal to the sum of the areas of all the triangles. If more triangles are constructed to fit in the circle, result will be more accurate. This is the method of exhaustion and it is similar to integral calculus body."

"I got it now. If a mathematician gets a calculus in the gallbladder, it is indeed painful to him or her. But how did the term get into to the vocabulary of mathematics?" I asked.

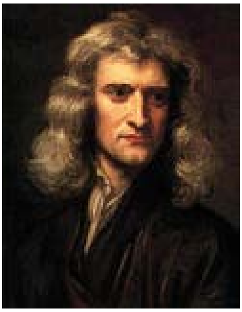
"It is said that about 15 BC, the Roman architect and engineer Vitruvius mounted a large wheel of known circumference in a small frame. When it was pushed along the ground by hand, it automatically dropped a pebble into a container at each revolution, giving a measure of the distance travelled. It was, in effect, the first odometer. So the calculus or pebble was used as a tool for counting process, which was the origin for the word 'calculate' in mathematics. This term was also picked up by mathematician later to describe a special branch of mathematics: Calculus."

Gottfried Leibniz (1646-1716)

"I have heard about calculus, but I don't know anything about this. Uncle, please tell me more about calculus."

"Calculus is a mathematical technique, mainly developed in the seventeenth century. It is a very powerful technique that has profound impact on mathematics."

"Who developed the concepts and methodologies of calculus?" I asked.



Isaac Newton 1642 - 1727

"The German mathematician Gottfried Wilhelm Leibniz and the English physicist Sir Isaac Newton independently developed calculus. However, an idea similar to calculus was conceptualised much earlier." Uncle replied.

"Please tell me how calculus was conceptualised," I was eager to know.

"During the third century BC, the Greek mathematician Archimedes used the method of exhaustion while trying to measure the area of a circle. He approximated a circle using many triangles and calculated the approximate area of a circle. This method is similar to what is known as integral calculus now."

"What happened after that?"

"In the 14th Century, the Indian mathematician Madhava of Sangamagrama and the Kerala School of Astronomy and Mathematics stated many rules on infinite series and approximations. These are considered to be similar to many components of calculus."

"This is great! India has a long tradition of mathematics. You told me that zero was first conceptualised in India. Ramanujan is one of the greatest mathematicians of the world and he contributed extensively in developing different branches of mathematics. It is no wonder that India also contributed to the development of the concept of calculus as well."

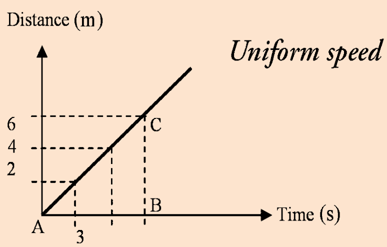
"You are right," uncle replied.

"I want to know more about calculus, please explain it to me."

"To put it in simple words, calculus is a mathematical technique which aids in solving two very practical types of problems. First, the varying rate of change, and secondly, measuring the area of irregular shape or volume of irregular three dimensional objects." Uncle said.

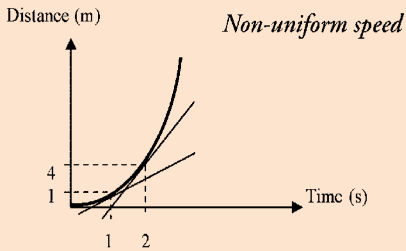
"What is the varying rate of change?" I was eagerly waiting for a breeze of mathematical ideas from my mathematician uncle.

"Consider a car is moving at uniform speed.



$$\text{Speed} = \frac{\textit{Distance}}{\textit{Time}} = \frac{2}{1} = \frac{4}{2} = \frac{6}{3} = 2 \text{ metre/second}$$

As ratio of distance to time is same at all time, the graph represents uniform speed.



Speed is different at different time. This is non-uniform speed.

Speed at any instance, say, at t = 2 sec, can be found by differentiation

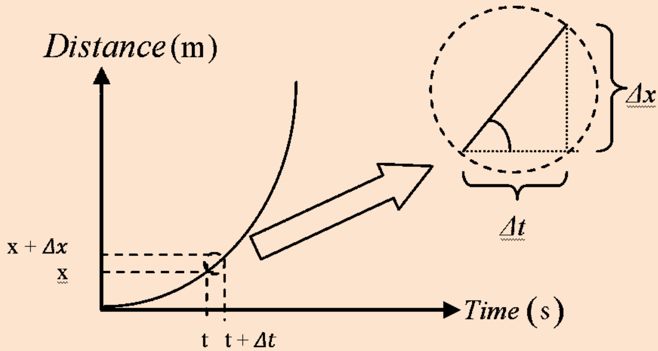
After one second it travels a distance of 2 metres, and after 2 seconds it travels a distance of 4 metres and so on. Now plot all points for each pair of observation (time and distance) on a graph taking the time in the x-axis and the distance travelled in the y-axis. Join all these points and you will get a straight line that graphically represents the distance travelled with respect to time. This straight line will make an angle with x–axis (time) (let's say, it is theta, θ). Tangent of this angle is called slope (shortened as *tan*θ). The slope represents the speed in a time-distance graph. However, if the car is accelerating then you will not get a straight line. Consider another case where the car travels 1 metre in 1 second, 4 metres in 2 seconds, 9 metres in 3 seconds and so on. If you plot all those paired observations (time and distance) on a graph now, you will get a curve. Tell me, how you will calculate the speed from this curve?" uncle asked.

"You said that the speed is changing – then how can I calculate the speed?"

"Good observation. As the speed is changing every moment, you have to calculate the speed at a given instance, say, the speed at $t = 2$ second – this is called the instantaneous speed. Differential calculus helps in calculating such type of problems,"

"But how do I calculate the instantaneous speed?" I wanted to know.

"Say, you want to calculate the speed of the car at $t = 2$ second.



At time t, let the distance travelled be x

After t, let in a very small time interval Δt, the car travels Δx distance. Δt is very small, hence the part of the curve is almost a straight line, as shown inside the dotted circle.

Hence, the speed at time t can be written as: $\frac{\Delta x}{\Delta t}$

This speed is the instantaneous speed (v) at time t and is represented as $\frac{dx}{dt}$, i.e. $v = \frac{dx}{dt}$

To do that, measure the distance travelled by the car in a very short time interval Δt, after 2 seconds. Let the distance travelled in time Δt be Δx. Now, the instantaneous speed is represented as: $\frac{\Delta x}{\Delta t}$. This is the basis of differential calculus."

"But uncle, how small is the time interval Δt?"

"It's a very good question. The Δt is smaller than the smallest time interval you can imagine or think of. In mathematical parlance, this is called *infinitesimally small*."

"Why do we need to take such a small time interval?" I wanted to know.

"A very small time interval Δt signifies that the change in speed during that interval will be insignificant, but it would travel a distance Δx. Hence the ratio of change in distance (Δx) to change in time (Δt) will be same at time t as well as at ($t + \Delta t$)."

"How can we measure such a small time interval and the distance travelled in that small time interval?"

"Differential calculus will help you in solving that. Let me explain. The distance travelled by a car can be represented as a function of time *t*, i.e. $x = f(t)$. For example, if the car is covering 1 metre in 1 second, 4 metres in 2 seconds, 9 metres in 3 seconds and so on, then this function f(*t*) is t^2 , i.e. $x = t^2$. If you now want to calculate speed at any instance, you can differentiate this function with respect to time *t*."

"Are there any rules for differentiation?" I wanted to know.

"Yes. There is a set of rules to calculate the derivative of a given function," uncle replied.

"I understood now. The derivative can be used to calculate the varying rate of change, for example, the non-uniform speed, which is the varying rate of change of distance with respect to time,' I commented.

"The derivative is useful to calculate the rate of change of any physical variable such as area, volume, pressure, force and so on. For example, using the derivative, one can calculate what will be the water pressure at certain depth of a dam having non-uniform cross-section."

"Uncle, you have mentioned two uses of calculus. You have explained the first one, i.e., how we can calculate the varying rate of change using the derivative. Please elaborate on the second use, i.e. how calculus helps us in calculating the area of an irregular shape or the volume of an irregular object."

"Calculation of area of any irregular shapes or volume of three-dimensional objects can be done by integration. Integral calculus is also called anti-derivative. In integral calculus, the term 'integral' is used to denote the summation of values. This is represented by an elongated 'S' symbol $\int$".

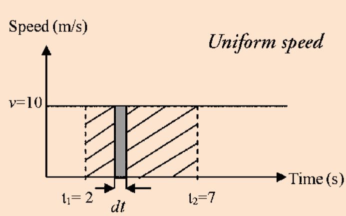
"That means differential and integral calculus are conceptually opposite to each other," I said.

"Well, you may say that. However, both are related. Integration deals with the summing up or the total of a variable. Referring back to our example on speed, we may be interested to know what is the total distance travelled by a car (with a variable speed) in a given time. This is possible to calculate only when you know the rate of change of speed at every instant. Please note that if the car has uniform speed, then you don't need calculus to find the distance travelled during a given interval."

"That's very interesting, please elaborate."

"Consider a car has uniform speed 10 m/s. Find out the distance travelled by it in an interval, say between the 2nd second and the 7th second"

"It's very easy – it is 10 × (7 – 2) = 50 metres," I replied.



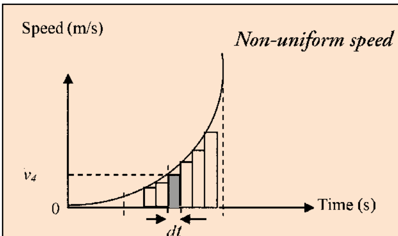
Distance travelled

$= v \times (t_2 - t_1)$

$= 10 \times (7 - 5)$

= 50 m

= Area under the graph



Distance travelled

= Area under the graph

= Sum of areas of all infinitesimal rectangles

$= v_1 \times dt + v_2 \times dt + ... + v_n \times dt$

$= \sum_{k=1}^{n} v_k \text{ x dt}$

"That's right. This signifies that in *speed-time* graph, the area under the graph (length × breadth = $v \times t$) will be the distance travelled by the car. Now, consider another car is having non-uniform speed. Consider speed is 1 metre/second when time is 1 sec; 4 metre/second when time is 2 sec; and 9 metre/second when time is 3 sec. In this case, the speed is represented by the function, $v = t^2$, v representing speed and t

representing time. Can you now calculate the distance travelled between 2 second and 7 second?" Uncle wanted to know.

“Uncle, in the previous example you said that the distance is a function of time and it may be represented by $x = t^2$ for a particular case. Now you are saying that the speed is a function of time and it may be represented as $v = t^2$. I am a bit confused now."

“Consider these two examples are for two different cars having different rate of change of speed. In first case, distance travelled was proportional to the square of time ($x = t^2$). In that case, the speed was proportional to time, not square of time, and I have explained how differential calculus will help in calculating the instantaneous speed from the equation $x = t^2$. In the second case, the speed is proportional to the square of time and we have to calculate distance travelled in a given interval using integral calculus. Note that I used the unit 'metre' in the first example to indicate the distance, and 'metre per second' in the second example to specify the speed."

“I got it now. Let me give it a try. The speed of the car at t = 2 sec is 4 metre/second, and at t = 7 sec is 49 metre/second. Can I calculate the distance travelled in the interval of 5 sec by considering the average speed i.e. $\frac{4+49}{2}$ = 26.5m/s?" I wanted to know.

“If you follow this strategy, it would be far from the actual distance travelled. Here the speed is changing at every moment, and therefore the distance travelled at every moment is different. Integration will help you in solving this kind of problem."

“Uncle, I am eager to know how to solve this problem."

“You might have understood now that the area under the *speed-time* graph will give you the total distance travelled by the car. If the area is a simple geometric shape, like a rectangle, you can simply multiply length (speed) with breadth (time). In case of non-uniform speed, similar concept is used in an infinitesimal scale. The entire area may be filled with infinitesimal rectangles, breadth of each is *dt*. It (*dt*) is the smallest possible time interval during which the change in speed is negligible. The length of each of these infinitesimal rectangles is the speed at that moment. For example, at the time t_4, the speed is v_4. Now the distance travelled during *dt* interval will be $v_4 \times dt$. If such distances are calculated for the entire time interval, say $t_1 = 2$ second to $t_n = 7$ second, you have to simply add all those infinitesimal rectangles. So the area of *n* infinitesimal rectangles can be calculated as: $v_1 \times dt + v_2 \times dt +$ .....+ $v_n \times dt$. This is the method of integration. Mathematically, a shorthand of writing the full expression of this sum is given by: $\sum_{k=1}^{n} v_k \times \mathrm{dt}$. The symbol $\sum$ signifies summation saying multiply each instance of *v* and *dt* and add all *n* such instances. This is also equivalent to symbol $\int$ in integral calculus."

“It is amazing! That means without integration we can't calculate the area of irregular shapes."

“You are right. Extending the same concept, the volume of any irregular shape can be calculated by taking the infinitesimal volume and summing them all."

“I used to think that calculus is very difficult to understand. Now I realised that it is very interesting and easy to understand."

“Indeed it is. Not only that; using calculus we are able to solve all real problems. Perfect geometric shapes or uniform speed do not happen in the nature. Almost everything in the nature is having non-uniform characteristic – growth of plants or animals, speed of any objects etc. etc. If you want to design a bridge, you have to consider the non-uniform load and speed. The shape of the bridge will not be a simple geometric shape; its stress and strain will vary on the differential load pattern at different points on the bridge. Only calculus can help in solving such real life problems."

“I have heard that extensive mathematical modelling is required for any space mission. I think that calculus is required for such kind of applications."

“You are right. The mathematical modelling and associated calculus helps us to understanding our universe, its past, present and the future. Mathematical modelling using calculus is also an essential component of quantum mechanics to know the micro world of atom and its nucleus, and many other branches of science. Calculus is used extensively by epidemiologists to model epidemic scenarios to explore how a disease would spread and what intervention strategies would help in combating the disease. In short, calculus plays the most vital role in explaining many concepts in physics, chemistry, biology, medicine and other subject areas. Its applications encompass the atoms to the universe, and thereby solving many real life problems."

“It's amazing! Uncle, thank you very much for introducing me to the world of calculus. The other day, I was reading a book on Tintin's adventure to the Moon. After exploring the exciting world of calculus, I had the same feeling as Professor Calculus exclaimed after his lunar exploration: what an adventure!"

Pondering over Probability

Last Sunday, I was very busy. Two of my friends had birthday celebrations on the same day. They are my best friends. I could not miss one for the other. So I attended both birthday parties. It was indeed a very enjoyable day celebrating their birthdays with other friends.

When I returned home, my uncle asked, 'My dear Googol, how were your parties?'

'Uncle, both parties were wonderful – I enjoyed them so much,' I could not resist my ecstatic expressions.

'That's very nice,' uncle smiled looking at my buoyant face.

'Uncle, it must be a very rare occasion that two of our classmates have the same birthday when we are only 50 friends in the class,' I was still reflecting on my moments of happy hours.

'On the contrary, it is very likely that two of your friends will have the same birth date. To be precise, there is a 97% chance that it would be so.' Uncle quipped.

'How can that be possible? Ignoring the leap years, there are 365 days in a year. How could it possibly be that two persons in a group of 50 will share the same birth date with such certainty?' I was completely perplexed.

'To understand that you have to get the concept of *probability*,' uncle replied.

'Please uncle, tell me more about it.' I pleaded with my uncle.

'Right, first tell me how you would mathematically arrange the following words in an ascending order: *coin*, *dice* and *card*.'

'I can tell you how they should appear in a dictionary. It should be *card* first, then *coin* and finally *dice*.' I tried to reason.

'Well, let me take it one by one. How many sides a coin has?' Uncle asked me again.

'Simple, a coin has two sides – *Head* and *Tail*.'

'Correct. Let's say, we plan an *experiment* which is tossing an unbiased coin once. In statistical jargon, we can say that the *outcome* Head (H) or Tail (T) is an *event*, and the collection of all such possible events is called the *sample space*.'

Sample space of tossing a coin once

'That means, if I *toss a coin once*, all possible events are only Head and Tail and therefore total number of possible events equals to *two*. Using the short name, I can say that the sample space will include H and T.'

'Exactly. The sample space is also denoted by the English or Greek letters like S, Ω or U (for universe), and the events within a sample space are sometime written with a curly bracket. So, for this particular experiment, we can write: $S = \{H, T\}$.' Uncle explained.

'I got it now.' I replied affirmatively.

'If you understand this, then we can very easily *estimate* the probability of an event.'

'What is a *probability*?' I interrupted.

'The term 'probability' has a very well-defined meaning in statistics. It is a measure of the expectation that an event will occur or a statement is true.'

'It seems a bit complicated,' I confessed.

'Don't worry. Let's take the example of tossing a coin.' Uncle took out a coin from his pocket and continued explaining me the coin tossing experiment. 'Imagine, we are doing an experiment of tossing a coin once. You know the sample space of all possible events from a single toss of a coin. Now, we can find out what is the probability of getting a 'Head' after a single toss. This can simply be obtained by: the number of times that the desired event i.e. 'H' is appearing in the sample space divided by the total number of events in the sample space.'

'Let me see if I understood it properly. The desired event is 'H'. The total number of events in a sample space for this experiment is *two* i.e. {H, T}. And, the desired event (H) is occurring in the sample space only *once*.'

'Very good. Backed with this information, we can now estimate the probability of getting a 'Head' in the experiment of tossing a coin once as: Pr(H) = 1/2.'

'It's very interesting. I got it now. So, the probability of a Tail in this experiment, or Pr(T) also equals to 1/2.'

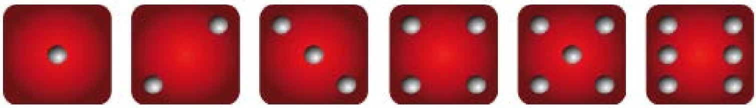
'You're right. Now, if you look at this concept carefully, you will notice another interesting property of probability that can be derived from this definition. By definition, the value of probability ranges between 0 and 1 inclusive. When it is zero, the desire event is *not* present in the sample space, and it is also called a *null event* or *impossible event*. If it is 1, all events in a sample space are of the desired event and this is absolute certainty. The higher the probability i.e. as it moves towards 1, the more certain we are that the event will occur.'

'So, Pr(H) or Pr(T) is just half-way of this range.'

'Yes, that's true. Well, in this context, let me tell you few more things. To estimate the probability in this way, it is assumed that each event in the sample space is *equally likely*. Statistically, it means that each event can occur with equal probability. Also, here we defined the sample space in a very simplified manner, but the mathematical definition of probability can extend to *infinite sample spaces*, and even an *uncountable sample space*. Anyway, we will not go into further details on this.'

'Uncle, please give me another example of probability.'

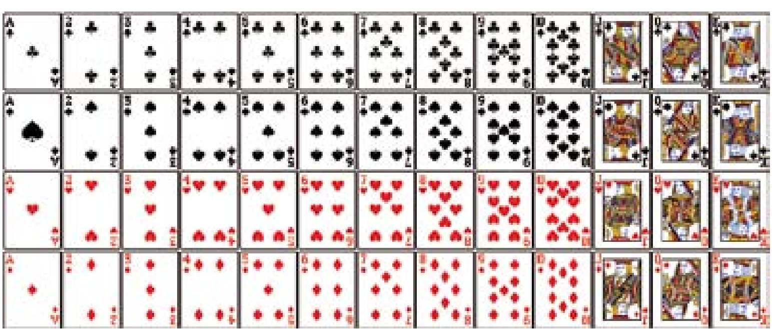
'Why don't you give a try on the *dice* example? If I roll a dice once, what is the probability of getting a 6?'



Sample space of rolling a dice once

'Oh yes, let me try it. There are *six* sides in a dice, and hence all possible events in one throw of a dice will include the sample space as {1, 2, 3, 4, 5, 6}. The desire event '6' happens only once, and so the Pr(6) = 1/6.'

'Well done. Let's talk about the *card*? Well, by card, I mean the playing cards. You may know that a deck of playing card has 13 cards of each suit (Ace or 1, 2 to 10, Jack, King and Queen). There are four suits: Clubs, Spades, Hearts and Diamonds. So, the total number of playing cards in a deck is 13 x 4 = 52.'



Sample space of a deck of playing cards

'So, the sample space includes a total of 52 events.'

'For some kinds of experiments, there may be two or more plausible sample spaces available. For example, when drawing a card from a standard deck of 52 playing cards, each one card can be an event and hence all possible events in the sample space is 52. However, one possibility for the sample space could be the rank (Ace through King ignoring the suit) and hence total events in the sample space are 13, where each event is appearing four times. Another possibility could be the suit (clubs, diamonds, hearts, or spades) where all possible events in the sample space are four, and each event is appearing 13 times. Now, if we are interested in each card as an event, could you please tell me what the probability of an Ace of Clubs is?'

'There is only one Ace of Clubs in all possible events in the sample space. So, Pr(Ace of Clubs) = 1/52.'

'That's nice Googol. Could you find out what the probability of an Ace of any suit is?'

'Let me try. There are a total of four Aces (of Clubs, Spades, Hearts and Diamonds) in the sample space. Then, Pr(Ace of any suit) = 4 / 52 or 1/13.'

'You've got this one too. Note here that you can solve this problem considering the sample space for the rank of cards (Ace through King) as mentioned before.'

'Yes uncle, I could get this now.'

'Let's make the game of probability slightly more complicated. We will go back to the dice experiment again. You now have a clear idea about sample space from a single throw of a dice. Can you figure out the probability of getting a 1 or 6?'

'Well, the sample space of a single throw of dice is: {1, 2, 3, 4, 5, 6}. Clearly, the Pr(1) = 1/6 and Pr(6) = 1/6. The desired events (1 or 6) are happening twice in the sample space. I guess, Pr(1 or 6) = 2/6. Am I right uncle?'

'Yes, you are thinking in the right direction. The desired event here indeed is either 1 or 6 which is occurring twice in the sample space. Another way to solve this problem is to add the Pr(1) and Pr(6).'

'Yes, that's true. Pr(1 or 6) = Pr(1) + Pr(6) = 1/6 + 1/6 = 2/6.'

‘Remember, you should not attempt *adding two or more probabilities* wherever you see them. If you do such operation indiscriminately, you may soon get a probability greater than 1 which is not possible (since the range of probability is zero to one). So, there is a rule for it. This is called the addition law of probability. You can simply add two probabilities if they are *mutually exclusive*. This is a statistical expression which means two or more events cannot occur at the same time, or the occurrence of an event excludes the occurrence of another event.’

‘Please, give me an example.’

‘For example, if you roll a dice once, events 1 and 6 can be termed as mutually exclusive events as both of them cannot appear in a single roll.’

‘Yes, I understand this now. I think that we can get other examples of mutually exclusive events from the roll of a dice. For example, getting 1 or 2 or any other numbers are also mutually exclusive.’

‘That’s correct, Googol. All events in a single roll of dice are mutually exclusive. To extend this concept further, the sum of probabilities of all mutually exclusive events in an experiment must add to 1. For example, rolling a dice once, the sum of *probabilities of* all mutually exclusive events in the entire sample space must add to one. In other words, Pr(1) + Pr(2) + Pr(3) + Pr(4) + Pr(5) + Pr(6) = 1.

‘I got it. For the single toss of a coin experiment, H – T are all possible mutually exclusive events in the entire sample space. Hence, Pr(H) + Pr(T) must be equal to 1.’

‘Yes Googol, you got it right.’

‘If I understood you correctly, you said that we cannot simply add probabilities when the events are *not* mutually exclusive.’

‘Yes, that’s true. If two events A and B are not mutually exclusive, then a simple addition will not do. This is because it will add the probability of occurrence of both A and B twice.

The occurrence of two events together may also be termed as *intersection* or *joint events*. In that case, the rule is to subtract the probability of intersection once. In other words, Pr(A *or* B) = Pr(A) + Pr(B) – Pr(A *and* B).'

'How do I get the Pr(A *and* B)?'

'We will talk about this soon. For the time being, note here that the word 'or' is indicative of using the law of addition. Well, applying the addition law of probability, could you please tell me the probability of getting an even number from rolling a dice?'

'There are three even numbers 2, 4 and 6 out of six numbers in the sample space. Therefore, the probability of getting an even number should be: Pr(2 or 4 or 6) = 3/6. Since these events are mutually exclusive, we can obtain this estimate by adding each probability: Pr(2 or 4 or 6) = 1/6 + 1/6 + 1/6 = 3/6.'

'That's right. Let's go back to our coin experiment again. Let's assume that the experiment now includes tossing a coin twice.'

'Is there a difference between tossing a coin twice and tossing two coins simultaneously?'

'It will not make any difference because the events are independent,. I'll explain this soon. Before that, could you now guess the sample space from such experiment i.e. tossing a coin twice or tossing two coins?'

'Well, the sample space for the first toss is: {H, T}, and so it is for the second toss. How can I get the sample space for the full experiment?'

'It's easy if you think about one event at a time. In the first toss,

Sample space of tossing two coins once (HH, HT, TH, TT) let's focus on the first event of the sample space, which is the 'H'. We know that the sample space in the second toss is: {H, T}. If we write the first and second toss together, the sample space can be expressed as: {HH, HT}. Similarly, with T, you will get: {TH, TT}. Putting those together, all possible events (or the sample space) for the experiment is: {HH, HT, TH, TT}.

'Yes, that's very interesting. I understand it now.'

'Well, then tell me the probability of getting one H from the first *and* one H from the second'

'I can clearly see how to estimate this. The desired event is HH and hence Pr(HH) = 1/4.'

'And, the probability of getting a 'H' in the first toss and a 'T' in the second toss...'

'The desired event is HT and so Pr(HT) = 1/4 again.'

'And, the probability of getting a 'H' and 'T' at any order, i.e. irrespective of any toss...'

'That means, both HT and TH are our desired events. Clearly, Pr(HT or TH) = 2/4, or we can add two probabilities since they are mutually exclusive, and we will get the same answer: 1/4 + 1/4.'

'You're doing well, Googol. Now, there is another interesting concept hidden in this experiment. Actually, tossing a coin twice can be termed as two *independent* events. Statistically, two or more events are independent events if the occurrence of one does not affect the probability of occurrence of the other.'

'Please explain this with an example.'

For example, if I toss a coin and if it comes as a 'Head', this outcome will not affect how the coin will behave if I toss the coin again. That means the Pr(H) in the first toss is 1/2, and the Pr(H) of the second toss will still be 1/2.'

'That means, rolling a dice twice or rolling two dice simultaneously will also result independent events since the occurrence of an event in the first roll will not affect the probability of events in the second roll.'

'That's correct. Now, here is another interesting law of probability. If two or more events are independent, then you can simply *multiply the probabilities* of each event to estimate the *joint probability* of all events. This is also called the *multiplication law of probability*. For example, for each toss of a coin, Pr(H) = Pr(T) = 1/2. If I toss a coin twice, the probability of getting one H from the first *and* one H from the second toss is: Pr(HH) = Pr(H from the first toss) x Pr(H from the second toss) = 1/2 x 1/2 = 1/4. We can do this multiplication, because occurrences of Head in the first and the second toss are independent.'

'And this is another way to get the results that we have seen before.'

'Yes, you are right. Here, the word 'and' generally indicates a possible use of the multiplication law of probability. Of course, as the experiment gets more complicated, you have to be very careful about identifying the events that are independent. For example, the probability of getting a 'H' and 'T' of any order from several tosses of a coin will include probabilities involving both independent and mutually exclusive events.'

'If the events are *not* independent, we can't then simply multiply two probabilities to get the joint probability.'

'Yes, that's true. Then the concept of *conditional probability* will come into play since the occurrence of one event is conditional or dependent upon another event. But I am not going into the intricacies of the conditional probability at this moment.'

'Okay, I will remember this note while applying the multiplication law of probability. Anyway, estimating probability with rolling two dice may be a little difficult problem. Uncle, could you please help me?' 'First, tell me if the word 'dice' is a singular or plural noun?' uncle asked me.

'I think that dice is a plural noun. But what's its singular form?'

'Historically, *dice* is the plural of *die*, but in modern Standard English, the word *dice* is used as both the singular and the plural. Anyway, here is the simplest problem. What's the probability of getting two sixes in rolling two dice?'

'Rolling two dice and getting a 6 in each of them are independent events. So, Pr(6 and 6) = 1/6 x 1/6 = 1/36.'

'Now, tell me if you roll two dice, what is the probability of getting the sum of two values as 7?'

'That's a bit complicated for me,' I confessed.

'There are a total of 36 combinations that two dices can produce. You can easily get it if you follow the same logic as we used for two coins problem. In other word, the sample space will include all these 36 combinations. Out of these 36 events, there are six events that will produce a 7. These are (6,1), (5,2), (4,3), (3,4), (2,5) and (1,6). Therefore probability of having a 7 is 6 out of 36, i.e. 1/6.'

'I got it now. Similarly, the probability of getting the sum as 11 is 2/36, as two combinations (6, 5) and (5, 6) will produce the sum as 11. The probability of getting the sum as 10 is 3/36, and so on."

'Here, is another tricky question. You just told me the probability of getting two sixes in a roll of two dice. Now, what is the probability of *not* getting two sixes in a roll of two dice?'

'Pr(6 and 6) = 1/36. But the question is of *non-occurrence* of these two events.' I was trying to figure it out.

'This is easy. Remember, the total probability of all mutually exclusive events in an experiment will be 1. With the help of this property, we can easily find out that the Pr(Not 6 and 6) = 1 – 1/36 = 35/36.'

'Yes, I understand it now – it simplifies the thing so much. For a single throw of dice, we can similarly say that: Pr(Not getting a 6) = 1 – Pr(6) = 1 – 1/6 = 5/6. Am I right?'

'You are absolutely right. In statistics parlance, this is also termed as *complementary event*. In probability theory, the complement of any event A means that the event A does not occur. It is expressed as A', A^c or $\bar{A}$. As we said earlier, Pr(A') = 1 – Pr(A). The events A and its complement are also mutually exclusive. Clearly, the sum of the probability of the complementary events equals to 1.'

'Uncle, I was always intrigued by probability. But it seems that if I know the basic concept of probability, this is indeed very fascinating.'

'Yes, whenever you get a problem with probability, frame your question clearly, try to decipher the logic correctly, and finally apply all rules of probability properly to get the answer. Otherwise, it's very easy to get confused!'

'Well, let's go back to the birthday problem. You mentioned that the chance that two friends in a group of 50 will share the same birth date is 97%. Could you please help me calculating the probability in this problem?'

'The problem can be solved using all the ideas that we discussed here like concepts of mutually exclusive events, independent events, complementary events, equally likely events etc. For simplicity, we will only consider the year as a non-leap year which has 365 days, and hence the total number of possible birthdays in the sample space is 365. We also assume that the 365 possible birthdays are equally likely. If Pr(B) is the probability of at least two friends in your class having the same birthday, it is generally simpler to calculate using the complementary probability Pr(B'), the probability of there not being any two friends having the same birthday. Let's put the spotlight on you first – you are the first person with a given birthday. We will unfold the logic by first excluding your birthday. Let's start.

The probability that the second friend in your class is *not* sharing the birthday with you is: $P(B_2') = 364/365$. This implies that the second friend's birthday should be in one of 364 days excluding your birthday.

Similarly, the probability that the third friend in your class is *not* sharing the birthday with you and friend 2 is: $P(B_3') = 363/365$.

The probability that the fourth friend in your class is *not* sharing the birthday with three of you is: $P(B_4') = 362/365$.

You can now see the pattern here. For 50 friends in a class, the probability that the fiftieth friend in your class is *not* sharing the birthday with other 49 friends is: $P(B'_{50}) = 316/365$.

Now, all these probabilities are independent events (the birthday of any given friend is independent of the birthday of other friends). Hence, we can multiply all these probabilities to obtain the probability that none of these 50 friends share a birthday.

$$P(B'_1) \times P(B'_2) \times P(B'_3) \times P(B'_4) \times \ldots\ldots \times P(B'_{50}) =$$

$$\frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \frac{362}{365} \times \ldots\ldots \times \frac{316}{365}$$

Denoting this overall probability as $Pr(B')$, the above expression can be written as:

$$P(B') = \frac{365 \times 364 \times 363 \times 362 \times \ldots\ldots \times 316}{365^{50}}$$

Or in general, for n friends, the simplified expression is:

$$P(B') = \frac{365 \times 364 \times 363 \times 362 \times \ldots\ldots \times (365 - n + 1)}{365^{n}}$$

Finally, we are interested in the event about the occurrence of at least one shared birthday. Then, using the rule of complementary probability, and because B and B' are the only two possibilities and they are also mutually exclusive, we can say that: $Pr(B) = 1 - Pr(B')$. Using some simple mathematical calculations, $Pr(B)$ can be obtained as 0.97, or the probability that two friends will share a birthday in a class of 50 is 97%. In fact, the probability is greater than 99% (almost a certainty) with 58 friends; and this probability is around 50.7% with only 23 friends.'

'Oh, that's very fascinating. That means in a football game, there is more than 50% chance that two persons among the players from teams, the referee and linesmen will have the same birthday.'

'From the probability sense, that's exactly the point!'

'I can now guess the answer to your words problem:

mathematically arranging the following words in ascending order: *coin*, *dice* and *card*.'

'Go on Googol.'

'Considering the total number of events in a sample space, the order in ascending order will be: *coin* (2), *dice* (6) and *card* (52). If we consider the probability of each mutually exclusive and equally likely event in the sample space, it will be: *card* (1/52), *dice* (1/6) and *coin* (1/2).'

'Fantastic, full marks to you, Googol.'

Srinivasa Ramanujan

Srinivasa Ramanujan is considered one of the greatest mathematicians of the twentieth century. Well-known mathematicians Professors G. H. Hardy and J.E. Littlewood compared Ramajuan's mathematical abilities and natural genius with all-time great mathematicians like Leonhard Euler, Carl Friedrich Gauss, and Karl Gustav Jacobi.

The influence of Ramanujan on number theory is without parallel in mathematics. His papers, problems, and letters would continue to captivate mathematicians for generations to come. He rediscovered a century of mathematics and made new discoveries.

Srinivasa Ramanujan Iyengar (best known as Srinivasa Ramanujan) was born on 22 December 1887, in Erode, about 400 km from Chennai (formerly Madras). Ramanujan's father Srinivasa Iyengar worked as an accountant for a cloth merchant. Ramanujan was the first child born to his mother Komalatammal.

Srinivasa Ramanujan
(1887 – 1920)

Ramanujan showed a strong inclination towards mathematics from early age and won numerous awards for his calculating skills in elementary school. He passed his primary examination in 1897 and then joined the Town High School.

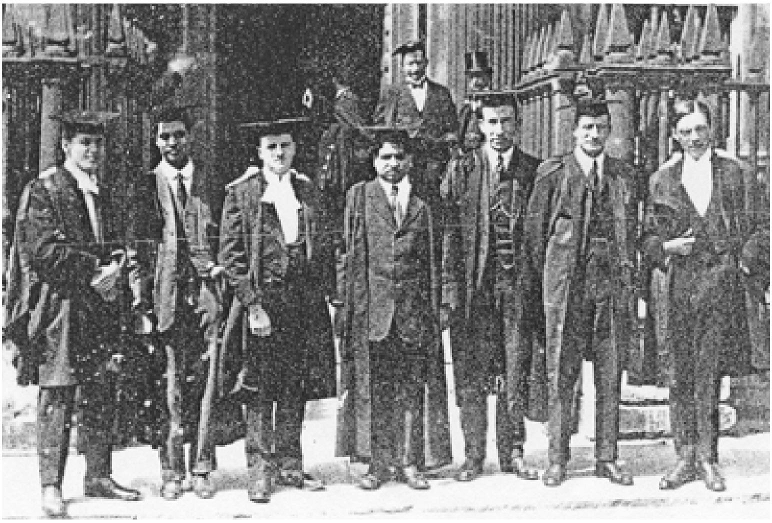
While at school, Ramanujan came across a book entitled *A Synopsis of Elementary Results in Pure and Applied Mathematics* by George Shoobridge Carr. This book had a great influence on Ramanujan's

G.H. Hardy (1877 – 1947)

career. G.H. Hardy (1877 – 1947), an eminent English mathematician wrote about the book: "He (Carr) is now completely forgotten, even in his college, except in so far as Ramanujan kept his name alive". Ramanujan solved all the problems in Carr's *Synopsis*. While working on the problems in the book, he discovered many other new formulae and provided results which were not there in the book. He jotted the results down in a notebook, which he showed to people he thought might be interested. Between 1903 and 1914 he had compiled three notebooks.

In 1904, Ramanujan entered Kumbakonam's Government College as F.A. student. He was awarded a scholarship. However, after school, Ramanujan's total concentration was focussed on mathematics and he neglected other subjects. As a result he failed and lost his scholarship. During 1906–1912 Ramanujan was constantly in search of a benefactor. Without a university degree it was very difficult for him to find a suitable job and had to struggle financially. Unfortunately he did not have anyone to direct him in his mathematical research. But that did not deter his passion for mathematics and he spent most of his time on mathematics. He noted down his results in his notebooks. These notebooks were his treasures. He looked for a job for livelihood and to support his parents and two brothers. He tutored a few students in mathematics. However, because of his unconventional methods, he was not considered to be a good teacher. Ramanujan's mother Komalatammal was on the lookout for a bride to get her eldest son married. On 14 July, 1909 Ramanujan was married to Janaki.

In 1910 Ramanujan met Professor V. Ramaswami Iyer, an ardent scholar of mathematics and founder of the Indian



Ramanujan (centre) with other scientists at Trinity College

Mathematical Society. After seeing the notebooks, Professor Ramaswami was convinced that Ramanujan was a gifted mathematician.

Ramanujan's earliest contribution was in the form of question/answer in the *Journal of the Indian Mathematical Society.* Ramanujan proposed 58 questions and their solutions during the period February 1911 to October 1911. The first full-length research paper of Ramanujan, entitled "Some properties of Bernoulli Numbers", appeared in the *Journal of the Indian Mathematical Society* in 1911.

In 1912, Ramanujan secured a job as a clerk in the accounts section of the Madras Port Trust. In the meantime his mathematical work caught the attention of other scholars who recognised his abilities. He was encouraged to contact English mathematicians in the hope that they would be able to assist him. Professor C.L.T. Griffith of Engineering College, Madras, forwarded some of Ramanujan's results on divergent series to Professor M.J.M. Hill of the University of London. Unfortunately, Professor Hill could not study the results in detail and suggested a book and gave advice as to how Ramanujan

On the occasion of 75th Birth anniversary of Ramanujan, the Indian Philately Association brought out a commemorative stamp in 1962

could get his paper published.

In 1913 Ramanujan wrote a letter to the famous English mathematician G.H.Hardy, who discussed Ramanujan's letter with his collaborator and friend, mathematician John Littlewood (1885–1977). After studying and discussing the letter, both realised that Ramanujan was a world-class mathematician and decided to bring Ramanujan to Cambridge. Ramanujan arrived in London on 14 April 1914. For the next five years, Ramanujan was associated with Hardy. Their collaboration represents the efforts of two great talents. Ramanujan was awarded the B.A. degree by research, in March 1916, for his work on highly composite numbers. He was the first Indian mathematician to be awarded the prestigious Fellowship of the Royal Society, in February 1918. Dr. P.C. Mahalanobis (1893–1972) was a student at King's College, Cambridge during that time and he became a good friend of Ramanujan.

The period of Ramanujan's stay in England almost overlapped with World War–I. During his five-year stay in Trinity College, Cambridge, Ramanujan published 21 research paper, five of which were in collaboration with Hardy. During this time Ramanujan also published short notes in the *Journal of the Indian Mathematical Society.*

After World War-I, Ramanujan returned to India in 1919. After his return from England his health deteriorated and his wife looked after him. Even during those months of prolonged illness Ramanujan kept on jotting down his mathematical calculations and results on sheets of paper. In January 1920, he wrote letter to Hardy and communicated his work on 'mock' theta function. Despite all the tender attention from his wife and the best medical attention from doctors, his health deteriorated. He breathed his last on 26 April 1920, at the age of 32.

After Ramanujan's death, Hardy tried systematic verification of Ramanujan's results from the second notebook. However, it was a daunting task and he persuaded the University of Madras to undertake the task. In 1931, the University of Madras requested Professor G.N. Watson to edit the notebooks in a suitable form for publication. This was a formidable task, since the notebooks contained over 300 theorems. Watson undertook the task of editing the notebooks with Professor B.M. Wilson. Unfortunately, Wilson passed away in 1935, virtually marking the end of the efforts to edit the notebooks.

The collected edition of Ramanujan's works was later edited by Hardy. The first edition of this book was published in 1927 by Cambridge University Press. This resulted in a flurry of research papers during the period 1928–38. In 1999, the American Mathematical society and London Mathematical Society reprinted the collected papers.

Much of Ramanujan's mathematics falls in the domain of number theory — the purest realm of mathematics. During his short lifetime, Ramanujan independently compiled nearly 3,900 results (mostly identities and equations). He stated results that were both original and highly unconventional, such as the Ramanujan prime and the Ramanujan theta function, and these have inspired a vast amount of further research in mathematics.

As Robart Kanigel says ".....few can say much about his work, and yet something in the story of his struggle for the chance to pursue his work on his own terms compels the imagination, leaving Ramanujan a symbol for genius, for the obstacles it faces, for the burdens it bears, for the pleasure it takes in its own existence."

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Moments in Mathematics

Mathematics is not about calculations or remembering complex rules to solve problems in the examination. It is also not about a monotonous and complex process for measuring, counting and accounting. We encounter mathematics in our everyday life in different forms. It also plays a predominant role in an overall development of the society. An understanding and appreciation of mathematics is therefore an essential life skill. While it helps to solve many real-life problems, it makes it possible to develop a logical thinking process.

The concepts of mathematics are unravelled through the conversation between young Googol and his uncle, who explains intricacies of mathematical issues and motivates him to ask more questions. The conversation is often witty and unfolds the mystical and the wonderful world of mathematics in an entertaining style. This book will take the reader to the beautiful and mesmerising world of mathematics.

About the author

Rintu Nath is a senior scientist with Vigyan Prasar, an organisation under the Department of Science and Technology. He is the Head of the Division of Information Systems of Vigyan Prasar. As a professional Engineer, he is interested in the field of data acquisition system and communication techniques. He is actively involved in developing and disseminating innovative open ended science experiments for joyful learning by students. He is also a popular science writer and regularly writes for a number of magazines and journals.

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